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Simplified variational principles for barotropic magnetohydrodynamics

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Variational principles for magnetohydrodynamics have been introduced by previous authors both in Lagrangian and Eulerian form. In this paper we introduce simpler Eulerian variational principles from which all the relevant equations of barotropic magnetohydrodynamics can be derived. The variational principle is given in terms of six independent functions for non-stationary barotropic flows with trivial topologies and three independent functions for stationary barotropic flows. This is less than the seven variables which appear in the standard equations of barotropic magnetohydrodynamics, which are the magnetic field $\boldsymbol{B}$ the velocity field $\boldsymbol{v}$ and the density ρ.

For non-trivial topologies it is necessary to assume that some of the variables introduced in the non-stationary formalism are non-single-valued. That is, it is necessary to introduce a number of branch cuts in order to define single-valued branches of the field variables. In turn, these cuts along with the six field variables constitute an extended number of dynamic variables. The number of cuts necessary depends on the flow. The relations between barotropic magnetohydrodynamics topological constants and the functions of the present formalism will be elucidated.

The equations obtained for non-stationary barotropic magnetohydrodynamics resemble the equations of Frenkel *et al.* (*Phys. Lett.* A, vol. 88, 1982, p. 461). The connection between the Hamiltonian formalism introduced in Frenkel *et al.* (1982) and the present Lagrangian formalism (with Eulerian variables) will be discussed.

1. Introduction

Variational principles for magnetohydrodynamics have been introduced by previous authors in Lagrangian and Eulerian form. Sturrock (1994) has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov & Moffatt (1995) in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three functions in addition to the seven variables which appear in the standard equations of magnetohydrodynamics, which are the magnetic field $\boldsymbol{B}$ the velocity field $\boldsymbol{v}$ and the density ρ. Kats (2003) has generalized Moffatt's work for compressible non-barotropic flows but without reducing the number of functions and the computational load. Moreover, Kats has shown that the variables he suggested

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can be utilized to describe the motion of arbitrary discontinuity surfaces (Kats & Kontorovich 1997; Kats 2001). Sakurai (1979) has introduced a two-function Eulerian variational principle for force-free magnetohydrodynamics and used it as the basis of a numerical scheme; his method is discussed in Sturrock (1994). A method of solving the equations for those two variables was introduced by Yang, Sturrock & Antiochos (1986).

In this work we will combine the Lagrangian of Sturrock (1994) with the Lagrangian of Sakurai (1979) to obtain an *Eulerian* Lagrangian principle which will depend on only six functions. The variational derivative of this Lagrangian will give us all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resemble the equations of Frenkel, Levich & Stilman (1982) (see also Zakharov & Kuznetsov 1997). The connection between the Hamiltonian formalism introduced in Frenkel *et al.* (1982) and the present Lagrangian formalism (with Eulerian variables) will be discussed. Furthermore, we will show that for stationary flows three functions will suffice in order to describe a Lagrangian principle for barotropic magnetohydrodynamics. The non-single-valuedness of the functions appearing in the reduced representation of non-stationary barotropic magnetohydrodynamics will be discussed in particular with connection to the topological invariants of magnetic helicities and cross-helicities. It will be shown how the conservation of cross-helicity can be easily generated using the Noether theorem and the variables introduced in this paper.

It should be emphasized that for non-trivial topologies it is necessary to assume that some of the variables introduced in the non stationary formalism are non-single-valued. That is, it is necessary to introduce a number of branch cuts in order to define single-valued branches of the field variables. In turn, these cuts along with the six field variables constitute an extended number of dynamic variables. The number of necessary cuts depends on the flow.

Owing to space limitations this paper is concerned only with barotropic magnetohydrodynamics. Variational principles of non-barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron (2000) in terms of 15 functions and Kats (2003) in terms of 20 functions. We suspect that this number can be somewhat reduced. Moreover, in a remarkable paper Kats (2004) (section IV, E) it is shown that there is a large symmetry group (gauge freedom) associated with the choice of those functions; this implies that the number of degrees of freedom can be reduced.

We anticipate applications of this study both to linear and nonlinear stability analysis of known barotropic magnetohydrodynamic configurations (Vladimirov, Moffatt & Ilin 1996, 1997, 1999; Alaguer *et al.* 1988) and for designing efficient numerical schemes for integrating the equations of magnetohydrodynamics (Yahalom, 2003; Yahalom & Pinhasi 2003; Yahalom, Pinhasi & Kopylenko 2005; Ophir *et al.* 2005).

The plan of this paper is as follows: first we introduce the standard notation and equations of barotropic magnetohydrodynamics. Next we review the Lagrangian variational principle of barotropic magnetohydrodynamics. This is followed by a review of the Eulerian variational principles of force-free magnetohydrodynamics. After those introductory sections we will present the six-function Eulerian variational principles for non-stationary magnetohydrodynamics. A derivation of the canonical momenta of the generalized coordinates appearing in the Lagrangian allows us to derive the system's Hamiltonian which resembles the Hamiltonian introduced by Frenkel *et al.* (1982). This is followed by the derivation of a variational principle for stationary magnetohydrodynamics. A discussion related to the magnetohydrodynamic topological constants concludes our paper.

2. The standard formulation of barotropic magnetohydrodynamics

2.1. *Basic equations*

The standard set of equations solved for barotropic magnetohydrodynamics are

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}), \tag{2.1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.2}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{2.3}$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla p(\rho) + \frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi}. \tag{2.4}$$

The following notation is utilized: $\partial/\partial t$ is the temporal partial derivative, $\mathrm{d}/\mathrm{d}t$ is the temporal material derivative and ∇ has its standard meaning in vector calculus. $\boldsymbol{B}$ is the magnetic field vector, $\boldsymbol{v}$ is the velocity field vector and ρ is the fluid density. Finally $p(\rho)$ is the pressure, which we assume depends on the density alone (barotropic case). The justification for these equations and the conditions under which they apply can be found in standard books on magnetohydrodynamics (see for example Sturrock 1994). Equation (2.1) describes the fact that the magnetic field lines are moving with the fluid elements ('frozen' magnetic field lines), (2.2) describes the fact that the magnetic field is solenoidal, (2.3) describes the conservation of mass and (2.4) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. The term

$$\boldsymbol{J} = \frac{\nabla \times \boldsymbol{B}}{4\pi} \tag{2.5}$$

is the electric current density which is not connected to any mass flow. The number of independent variables for which one needs to solve is seven ($\boldsymbol{v}$, $\boldsymbol{B}$, ρ) and the number of equations (2.1), (2.3), (2.4) is also seven. Notice that (2.2) is a condition on the initial $\boldsymbol{B}$ field and is satisfied automatically for any other time due to (2.1). Also notice that $p(\rho)$ is not a variable, rather it is a given function of ρ.

2.2. *Lagrangian variational principle of magnetohydrodynamics*

A Lagrangian variational principle for barotropic magnetohydrodynamics has been discussed by a number of authors (see for example Sturrock 1994) and an outline of this approach is given below. Consider the action:

$$\left.\begin{aligned} A &\equiv \int \mathscr{L}\, \mathrm{d}^3x\, \mathrm{d}t, \\ \mathscr{L} &\equiv \rho\left(\tfrac{1}{2}\boldsymbol{v}^2 - \varepsilon(\rho)\right) - \frac{\boldsymbol{B}^2}{8\pi}, \end{aligned}\right\} \tag{2.6}$$

in which $\varepsilon(\rho)$ is the specific internal energy. A variation in any quantity F for a fixed position $\boldsymbol{r}$ is denoted as δF, hence:

$$\left.\begin{aligned} \delta A &= \int \delta\mathscr{L}\, \mathrm{d}^3x\, \mathrm{d}t, \\ \delta\mathscr{L} &= \delta\rho\left(\tfrac{1}{2}\boldsymbol{v}^2 - w(\rho)\right) + \rho\boldsymbol{v}\cdot\delta\boldsymbol{v} - \frac{\boldsymbol{B}\cdot\delta\boldsymbol{B}}{4\pi}, \end{aligned}\right\} \tag{2.7}$$

in which $w = \partial(\varepsilon\rho)/\partial\rho$ is the specific enthalpy.

A change in the position of a fluid element located at a position $\boldsymbol{r}$ at time t is given by $\boldsymbol{\xi}(\boldsymbol{r}, t)$. A mass-conserving variation of ρ takes the form

$$\delta\rho = -\nabla\cdot(\rho\boldsymbol{\xi}) \tag{2.8}$$

and a magnetic-flux-conserving variation takes the form

$$\delta\boldsymbol{B} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}). \tag{2.9}$$

A change involving a local variation coupled with a change of element position of the quantity F is given by

$$\Delta F = \delta F + (\boldsymbol{\xi}\cdot\nabla)F, \tag{2.10}$$

hence

$$\Delta\boldsymbol{v} = \delta\boldsymbol{v} + (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}. \tag{2.11}$$

However, since

$$\Delta\boldsymbol{v} = \Delta\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\Delta\boldsymbol{r}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t}, \tag{2.12}$$

we obtain

$$\delta\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\xi}}{\mathrm{d}t} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v} = \frac{\partial\boldsymbol{\xi}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{v}. \tag{2.13}$$

Introducing the result of (2.8), (2.9), (2.13) into (2.7) and integrating by parts we arrive at the result

$$\begin{aligned}\delta A = &\int \mathrm{d}^3x\rho\boldsymbol{v}\cdot\boldsymbol{\xi}|_{t_0}^{t_1}\\ &+\int \mathrm{d}t\biggl\{\oint \mathrm{d}\boldsymbol{S}\cdot\left[-\rho\boldsymbol{\xi}\left(\tfrac{1}{2}\boldsymbol{v}^2 - w(\rho)\right) + \rho\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\xi}) + \frac{1}{4\pi}\boldsymbol{B}\times(\boldsymbol{\xi}\times\boldsymbol{B})\right]\\ &+\int \mathrm{d}^3x\boldsymbol{\xi}\cdot\left[-\rho\nabla w - \frac{\partial(\rho\boldsymbol{v})}{\partial t} - \frac{\partial(\rho\boldsymbol{v}v_k)}{\partial x_k} - \frac{1}{4\pi}\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]\biggr\},\end{aligned} \tag{2.14}$$

in which a summation convention is assumed. Taking into account the continuity (2.3) we obtain

$$\begin{aligned}\delta A = &\int \mathrm{d}^3x\rho\boldsymbol{v}\cdot\boldsymbol{\xi}|_{t_0}^{t_1}\\ &+\int \mathrm{d}t\biggl\{\oint \mathrm{d}\boldsymbol{S}\cdot\left[-\rho\boldsymbol{\xi}\left(\tfrac{1}{2}\boldsymbol{v}^2 - w(\rho)\right) + \rho\boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{\xi}) + \frac{1}{4\pi}\boldsymbol{B}\times(\boldsymbol{\xi}\times\boldsymbol{B})\right]\\ &+\int \mathrm{d}^3x\boldsymbol{\xi}\cdot\left[-\rho\nabla w - \rho\frac{\partial\boldsymbol{v}}{\partial t} - \rho(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \frac{1}{4\pi}\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]\biggr\},\end{aligned} \tag{2.15}$$

hence we see that if $\delta A = 0$ for a $\boldsymbol{\xi}$ vanishing at the initial and final times and on the surface of the domain but otherwise arbitrary then Euler's (2.4) is satisfied (taking into account that in the barotropic case $\nabla w = \nabla p/\rho$). Note that the vanishing of $\boldsymbol{\xi}$ on the surface of the domain is only a sufficient condition for Euler's equations to be satisfied. It is certainly not a necessary condition. In scenarios such that the boundary is impermeable and perfectly conducting, it suffices to assume that only the normal component of $\boldsymbol{\xi}$ is zero at the boundary.

Although the variational principle does give us the correct dynamical equation for an arbitrary $\boldsymbol{\xi}$, it has the following deficiencies:

(*a*) Although $\boldsymbol{\xi}$ is quite arbitrary the variations of $\delta\rho$ and $\delta\boldsymbol{B}$ are not. They are defined by the conditions given in (2.8) and (2.9). This property is not useful for numerical schemes since $\boldsymbol{\xi}$ must be a small quantity.

(*b*) Only (2.4) is derived from the variational principle; the other equations that are needed: (2.1), (2.2) and (2.3), are separate assumptions. Moreover (2.3) is needed in order to derive Euler's (2.4) from the variational principle. All this makes the variational principle less useful.

What is desired is a variational principle from which all equations of motion can be derived and for which no assumptions on the variations are needed; this will be discussed in the following sections.

Note also two recent interesting papers by Prix (2004, 2005) which discuss the implications of a time shift τ in addition to the spatial shift $\boldsymbol{\xi}(\boldsymbol{r}, t)$ and also consider the case of multi-fluid magnetohydrodynamics.

3. Sakurai's variational principle of force-free magnetohydrodynamics

Force-free magnetohydrodynamics is concerned with the case that both the pressure and inertial terms in Euler (2.4) are physically insignificant. Hence the Euler equations can be written in the form

$$\frac{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} = \boldsymbol{J} \times \boldsymbol{B} = 0. \tag{3.1}$$

In order to describe force-free fields Sakurai (1979) has proposed representing the magnetic field in the following form:

$$\boldsymbol{B} = \nabla\chi \times \nabla\eta. \tag{3.2}$$

Hence $\boldsymbol{B}$ is orthogonal both to $\nabla\chi$ and $\nabla\eta$. A similar representation was suggested by Dungey (1958) but not in the context of variational analysis. Frenkel *et al.* (1982) discuss the validity of the above representation and have concluded that for a vector field in the Euclidean space $\mathscr{R}^3$ it does always exist locally but not always globally. Also note that either χ or η (or both) can be non-single-valued functions (see Frenkal *et al.* 1982, equation 20).

Both χ and η are Clebsch-type comoving scalar fields satisfying the equations

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}\eta}{\mathrm{d}t} = 0. \tag{3.3}$$

It can be easily shown that provided that $\boldsymbol{B}$ is in the form given in (3.2), and (3.3) is satisfied, then both (2.1) and (2.2) are satisfied. Since according to (3.1) both $\nabla \times \boldsymbol{B}$ and $\boldsymbol{B}$ are parallel it follows that (3.1) can be written as

$$\boldsymbol{J} \cdot \nabla\chi = 0, \qquad \boldsymbol{J} \cdot \nabla\eta = 0. \tag{3.4}$$

Sakurai (1979) has introduced an action principle from which (3.4) can be derived:

$$\left.\begin{aligned} A_S &\equiv \int \mathscr{L}_S \, \mathrm{d}^3x \, \mathrm{d}t, \\ \mathscr{L}_S &\equiv \frac{\boldsymbol{B}^2}{8\pi} = \frac{(\nabla\chi \times \nabla\eta)^2}{8\pi}. \end{aligned}\right\} \tag{3.5}$$

Taking the variation of (3.5) we obtain

$$\left.\begin{aligned} \delta A_S &= \int \delta\mathscr{L}_S \, \mathrm{d}^3x \, \mathrm{d}t, \\ \delta\mathscr{L}_S &= \frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot}(\nabla\delta\chi \times \nabla\eta + \nabla\chi \times \nabla\delta\eta). \end{aligned}\right\} \tag{3.6}$$

Integrating by parts and using the theorem of Gauss one obtains the result

$$\begin{aligned} \delta A_S = \oint \mathrm{d}\boldsymbol{S}\boldsymbol{\cdot}\left[(\delta\chi\nabla\eta - \delta\eta\nabla\chi) \times \frac{\boldsymbol{B}}{4\pi}\right] + \int \mathrm{d}\boldsymbol{\Sigma}\boldsymbol{\cdot}\left[([\delta\chi]\nabla\eta - [\delta\eta]\nabla\chi) \times \frac{\boldsymbol{B}}{4\pi}\right] \\ + \int \mathrm{d}^3x[\delta\chi(\nabla\eta\boldsymbol{\cdot}\boldsymbol{J}) - \delta\eta(\nabla\chi\boldsymbol{\cdot}\boldsymbol{J})] \end{aligned} \tag{3.7}$$

in which $\int \mathrm{d}\boldsymbol{\Sigma}$ represents an integral along the cut and $[\delta f]$ represents the discontinuity of the variations of non-single-valued functions. We introduce cuts in the domain because we are not sure at this stage whether the χ and η functions are single-valued or multiple-valued. We shall show later that χ can be defined as a single-valued function while η can be either single-valued or non-single-valued. In the latter case a sufficient condition for the 'cut' term to vanish is the usage of a single-valued $\delta\eta$, that is we may vary η only using single-valued variations. Hence if $\delta A_S = 0$ for arbitrary variation $\delta\chi, \delta\eta$ that vanish on the boundary of the domain (including the cut) one recovers the force-free Euler equations (3.4).

Although this approach is better than the one described in (2.6) in the previous section in the sense that the form of the variations $\delta\chi, \delta\eta$ is not constrained, it has some limitations as follows:

(*a*) Sakurai's approach by design is only meant to deal with force-free magnetohydrodynamics; for more general magnetohydrodynamics it is not adequate.

(*b*) Sakurai's action given by (3.5) contains all the relevant physical equations only if the configuration is static ($\boldsymbol{v}=0$). If the configuration is not static one needs to supply an additional two equations (3.3) to the variational principle.

4. Simplified variational principle of non-stationary barotropic magnetohydrodynamics

In this section we will combine the approaches described in the previous sections in order to obtain a variational principle of non-stationary barotropic magnetohydrodynamics such that all the relevant barotropic magnetohydrodynamic equations can be derived from using unconstrained variations. The approach is based on a method first introduced by Seliger & Whitham (1968). Consider the action

$$\left.\begin{aligned} & A \equiv \int \mathscr{L} \, \mathrm{d}^3x \, \mathrm{d}t, \\ & \mathscr{L} \equiv \mathscr{L}_1 + \mathscr{L}_2, \quad \mathscr{L}_1 \equiv \rho\left(\tfrac{1}{2}\boldsymbol{v}^2 - \varepsilon(\rho)\right) + \frac{\boldsymbol{B}^2}{8\pi}, \\ & \mathscr{L}_2 \equiv \nu\left[\frac{\partial\rho}{\partial t} + \nabla\boldsymbol{\cdot}(\rho\boldsymbol{v})\right] - \rho\alpha\frac{\mathrm{d}\chi}{\mathrm{d}t} - \rho\beta\frac{\mathrm{d}\eta}{\mathrm{d}t} - \frac{\boldsymbol{B}}{4\pi}\boldsymbol{\cdot}(\nabla\chi \times \nabla\eta). \end{aligned}\right\} \tag{4.1}$$

Obviously ν, α, β are Lagrange multipliers which were inserted in such a way that the variational principle will yield the following equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \qquad \rho \frac{\mathrm{d}\chi}{\mathrm{d}t} = 0, \qquad \rho \frac{\mathrm{d}\eta}{\mathrm{d}t} = 0. \tag{4.2}$$

It is not assumed that ν, α, β are single-valued. Provided ρ is not null these are just the continuity equation (2.3) and the conditions that Sakurai's functions are comoving as in (3.3). Taking the variational derivative with respect to $\boldsymbol{B}$ we see that

$$\boldsymbol{B} = \hat{\boldsymbol{B}} \equiv \nabla \chi \times \nabla \eta. \tag{4.3}$$

Hence $\boldsymbol{B}$ is in Sakurai's form and satisfies (2.2). By virtue of (4.2) we see that $\boldsymbol{B}$ must also satisfy (2.1). For the time being we have showed that all the equations of barotropic magnetohydrodynamics can be obtained from the above variational principle except Euler's equations. We will now show that Euler's equations can be derived from the above variational principle as well. Let us take an arbitrary variational derivative of the above action with respect to $\boldsymbol{v}$; this will result in

$$\delta_{\boldsymbol{v}} A = \int \mathrm{d}^3x \, \mathrm{d}t \rho \delta \boldsymbol{v} \cdot [\boldsymbol{v} - \nabla \nu - \alpha \nabla \chi - \beta \nabla \eta] + \oint \mathrm{d}\boldsymbol{S} \cdot \delta \boldsymbol{v} \rho \nu + \int \mathrm{d}\boldsymbol{\Sigma} \cdot \delta \boldsymbol{v} \rho [\nu]. \tag{4.4}$$

The integral $\oint \mathrm{d}\boldsymbol{S} \cdot \delta \boldsymbol{v} \rho \nu$ vanishes in many physical scenarios. In the case of astrophysical flows it will vanish since $\rho = 0$ on the flow boundary; in the case of a fluid contained in a vessel no-flux boundary conditions $\delta \boldsymbol{v} \cdot \hat{\boldsymbol{n}} = 0$ are induced ($\hat{\boldsymbol{n}}$ is a unit vector normal to the boundary). The surface integral $\int \mathrm{d}\boldsymbol{\Sigma}$ on the cut of ν vanishes in the case that the flow has zero cross-helicity (see § 7) since in this case ν is single-valued and $[\nu] = 0$. In the case that that the flow has non-zero cross-helicity, ν is not single-valued (see § 7); in this case only a Kutta-type velocity perturbation (Yahalom *et al.* 2005) is parallel to the cut will cause the cut integral to vanish.

Provided that the surface integrals do vanish and that $\delta_{\boldsymbol{v}} A = 0$ for an arbitrary velocity perturbation we see that $\boldsymbol{v}$ must have the following form:

$$\boldsymbol{v} = \hat{\boldsymbol{v}} \equiv \nabla \nu + \alpha \nabla \chi + \beta \nabla \eta. \tag{4.5}$$

Let us now take the variational derivative with respect to the density ρ; we obtain

$$\delta_\rho A = \int \mathrm{d}^3x \, \mathrm{d}t \delta \rho \left[\tfrac{1}{2} \boldsymbol{v}^2 - w - \frac{\partial \nu}{\partial t} - \boldsymbol{v} \cdot \nabla \nu \right] + \oint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{v} \delta \rho \nu + \int \mathrm{d}\boldsymbol{\Sigma} \cdot \boldsymbol{v} \delta \rho [\nu] + \int \mathrm{d}^3x \nu \delta \rho |_{t_0}^{t_1}. \tag{4.6}$$

Hence provided that $\oint \mathrm{d}\boldsymbol{S} \cdot \boldsymbol{v} \delta \rho \nu$ vanishes on the boundary of the domain and $\int \mathrm{d}\boldsymbol{\Sigma} \cdot \boldsymbol{v} \delta \rho [\nu]$ vanishes on the cut of ν, in the case that ν is not single-valued (which entails either a Kutta-type condition for the velocity or a vanishing density perturbation on the cut) and $\partial \rho$ vanishes at initial and final times, the following equation must be satisfied:

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \tfrac{1}{2} \boldsymbol{v}^2 - w. \tag{4.7}$$

Finally we have to calculate the variation with respect to both χ and η; this will lead us to the following results:

$$\delta_\chi A = \int d^3x\, dt \delta\chi \left[\frac{\partial(\rho\alpha)}{\partial t} + \nabla\cdot(\rho\alpha\boldsymbol{v}) - \nabla\eta\cdot\boldsymbol{J}\right] + \oint d\boldsymbol{S}\cdot\left[\frac{\boldsymbol{B}}{4\pi}\times\nabla\eta - \boldsymbol{v}\rho\alpha\right]\delta\chi$$
$$+ \int d\boldsymbol{\Sigma}\cdot\left[\frac{\boldsymbol{B}}{4\pi}\times\nabla\eta - \boldsymbol{v}\rho\alpha\right][\delta\chi] - \int d^3x\rho\alpha\delta\chi|_{t_0}^{t_1}, \tag{4.8}$$

$$\delta_\eta A = \int d^3x\, dt \delta\eta \left[\frac{\partial(\rho\beta)}{\partial t} + \nabla\cdot(\rho\beta\boldsymbol{v}) + \nabla\chi\cdot\boldsymbol{J}\right] + \oint d\boldsymbol{S}\cdot\left[\nabla\chi\times\frac{\boldsymbol{B}}{4\pi} - \boldsymbol{v}\rho\beta\right]\delta\eta$$
$$+ \int d\boldsymbol{\Sigma}\cdot\left[\nabla\chi\times\frac{\boldsymbol{B}}{4\pi} - \boldsymbol{v}\rho\beta\right][\delta\eta] - \int d^3x\rho\beta\delta\eta|_{t_0}^{t_1}. \tag{4.9}$$

Provided that the correct temporal and boundary conditions are met with respect to the variations $\delta\chi$ and $\delta\eta$ on the domain boundary and on the cuts in the case that some (or all) of the relevant functions are non-single-valued, we obtain the following set of equations:

$$\frac{d\alpha}{dt} = \frac{\nabla\eta\cdot\boldsymbol{J}}{\rho}, \qquad \frac{d\beta}{dt} = -\frac{\nabla\chi\cdot\boldsymbol{J}}{\rho}, \tag{4.10}$$

in which the continuity equation (2.3) was taken into account. By correct temporal conditions we mean that both $\delta\eta$ and $\delta\chi$ vanish at initial and final times. As boundary conditions which are sufficient to make the boundary term vanish we can consider the case that the boundary is at infinity and both $\boldsymbol{B}$ and ρ vanish. Another possibility is that the boundary is impermeable and perfectly conducting. A sufficient condition for the integral over the 'cuts' to vanish is to use variations $\delta\eta$ and $\delta\chi$ which are single-valued. It will be shown later that χ can always be taken to be single-valued, hence taking $\delta\chi$ to be single-valued is no restriction at all. In some topologies η is not single-valued and in those cases a single-valued restriction on $\delta\eta$ is sufficient to make the cut term null.

4.1. *Euler's equations*

We shall now show that a velocity field given by (4.5), such that the equations for $\alpha, \beta, \chi, \eta, \nu$ satisfy the corresponding equations (4.2), (4.7), (4.10) must satisfy Euler's equations. Let us calculate the material derivative of $\boldsymbol{v}$:

$$\frac{d\boldsymbol{v}}{dt} = \frac{d\nabla\nu}{dt} + \frac{d\alpha}{dt}\nabla\chi + \alpha\frac{d\nabla\chi}{dt} + \frac{d\beta}{dt}\nabla\eta + \beta\frac{d\nabla\eta}{dt}. \tag{4.11}$$

It can be easily shown that

$$\left.\begin{aligned}
\frac{d\nabla\nu}{dt} &= \nabla\frac{d\nu}{dt} - \nabla v_k\frac{\partial\nu}{\partial x_k} = \nabla\left(\tfrac{1}{2}\boldsymbol{v}^2 - w\right) - \nabla v_k\frac{\partial\nu}{\partial x_k},\\
\frac{d\nabla\eta}{dt} &= \nabla\frac{d\eta}{dt} - \nabla v_k\frac{\partial\eta}{\partial x_k} = -\nabla v_k\frac{\partial\eta}{\partial x_k},\\
\frac{d\nabla\chi}{dt} &= \nabla\frac{d\chi}{dt} - \nabla v_k\frac{\partial\chi}{\partial x_k} = -\nabla v_k\frac{\partial\chi}{\partial x_k},
\end{aligned}\right\} \tag{4.12}$$

in which x_k is a Cartesian coordinate and a summation convention is assumed. Equations (4.2), (4.7) were used in the above derivation. Inserting the result from

(4.12), (4.10) into (4.11) yields

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -\nabla v_k\left(\frac{\partial \nu}{\partial x_k}+\alpha\frac{\partial \chi}{\partial x_k}+\beta\frac{\partial \eta}{\partial x_k}\right)+\nabla\left(\tfrac{1}{2}\boldsymbol{v}^2-w\right)\\ &\quad+\frac{1}{\rho}((\nabla\eta\cdot\boldsymbol{J})\nabla\chi-(\nabla\chi\cdot\boldsymbol{J})\nabla\eta)\\ &= -\nabla v_k v_k+\nabla\left(\left(\tfrac{1}{2}\boldsymbol{v}^2-w\right)+\frac{1}{\rho}\boldsymbol{J}\times(\nabla\chi\times\nabla\eta)\right)\\ &= -\frac{\nabla p}{\rho}+\frac{1}{\rho}\boldsymbol{J}\times\boldsymbol{B}, \end{aligned}\tag{4.13}$$

in which we have used both (4.5) and (4.3). This of course proves that the barotropic Euler equations can be derived from the action given in (4.1) and hence all the equations of barotropic magnetohydrodynamics can be derived from the above action without restricting the variations in any way except on the relevant boundaries and cuts. The reader should take into account that the topology of the magnetohydrodynamic flow is conserved, hence cuts must be introduced into the calculation as initial conditions.

4.2. *Simplified action*

One might argue here that the paper is misleading. We have declared that we shall present a simplified action for barotropic magnetohydrodynamics but instead have added five more functions $\alpha, \beta, \chi, \eta, \nu$ to the standard set $\boldsymbol{B}, \boldsymbol{v}, \rho$. In the following we will show that this is not so and the action given in (4.1) in a form suitable for a pedagogic presentation can indeed be simplified. It is easy to show that the Lagrangian density appearing in (4.1) can be written in the form

$$\begin{aligned}\mathscr{L} &= -\rho\left[\frac{\partial \nu}{\partial t}+\alpha\frac{\partial \chi}{\partial t}+\beta\frac{\partial \eta}{\partial t}+\varepsilon(\rho)\right]+\tfrac{1}{2}\rho[(\boldsymbol{v}-\hat{\boldsymbol{v}})^2-(\hat{\boldsymbol{v}})^2]\\ &\quad+\frac{1}{8\pi}[(\boldsymbol{B}-\hat{\boldsymbol{B}})^2-(\hat{\boldsymbol{B}})^2]+\frac{\partial(\nu\rho)}{\partial t}+\nabla\cdot(\nu\rho\boldsymbol{v}),\end{aligned}\tag{4.14}$$

in which $\hat{\boldsymbol{v}}$ is a shorthand notation for $\nabla\nu+\alpha\nabla\chi+\beta\nabla\eta$ (see (4.5)) and $\hat{\boldsymbol{B}}$ is a shorthand notation for $\nabla\chi\times\nabla\eta$ (see (4.3)). Thus $\mathscr{L}$ has four contributions:

$$\left.\begin{aligned}&\mathscr{L}=\hat{\mathscr{L}}+\mathscr{L}_{\boldsymbol{v}}+\mathscr{L}_{\boldsymbol{B}}+\mathscr{L}_{boundary},\\ &\hat{\mathscr{L}}\equiv-\rho\left[\frac{\partial \nu}{\partial t}+\alpha\frac{\partial \chi}{\partial t}+\beta\frac{\partial \eta}{\partial t}+\varepsilon(\rho)+\tfrac{1}{2}(\nabla\nu+\alpha\nabla\chi+\beta\nabla\eta)^2\right]-\frac{1}{8\pi}(\nabla\chi\times\nabla\eta)^2\\ &\mathscr{L}_{\boldsymbol{v}}\equiv\tfrac{1}{2}\rho(\boldsymbol{v}-\hat{\boldsymbol{v}})^2,\quad \mathscr{L}_{\boldsymbol{B}}\equiv\frac{1}{8\pi}(\boldsymbol{B}-\hat{\boldsymbol{B}})^2,\\ &\mathscr{L}_{boundary}\equiv\frac{\partial(\nu\rho)}{\partial t}+\nabla\cdot(\nu\rho\boldsymbol{v}).\end{aligned}\right\}\tag{4.15}$$

The only term containing $\boldsymbol{v}$ is $\mathscr{L}_{\boldsymbol{v}}$†, and it can easily be seen that this term will lead, after we nullify the variational derivative with respect to $\boldsymbol{v}$, to (4.5) but will otherwise have no contribution to other variational derivatives. Similarly the only term containing $\boldsymbol{B}$ is $\mathscr{L}_{\boldsymbol{B}}$ and it can easily be seen that this term will lead, after

† $\mathscr{L}_{boundary}$ also depends on $\boldsymbol{v}$ but being a boundary term in space and time it does not contribute to the derived equations.

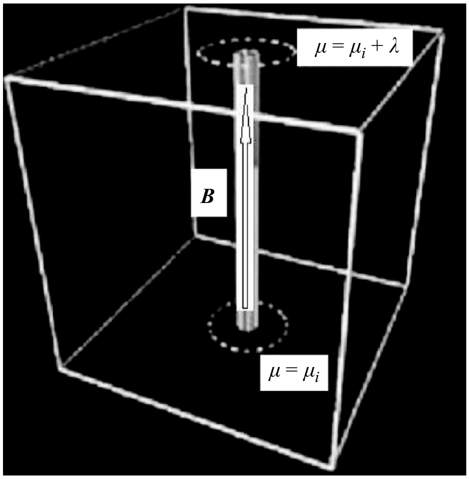

FIGURE 1. A thin tube surrounding a magnetic field line.

we nullify the variational derivative, to (4.3) but will have no contribution to other variational derivatives. Also notice that the term $\mathscr{L}_{boundary}$ contains only complete partial derivatives and thus cannot contribute to the equations although it can change the boundary conditions. Hence we see that (4.2), (4.7) and (4.10) can be derived using the Lagrangian density $\hat{\mathscr{L}}[\alpha, \beta, \chi, \eta, \nu, \rho]$ in which $\hat{\boldsymbol{v}}$ replaces $\boldsymbol{v}$ and $\hat{\boldsymbol{B}}$ replaces $\boldsymbol{B}$ in the relevant equations. Furthermore, after integrating the six equations (4.2), (4.7), (4.10) we can insert the potentials $\alpha, \beta, \chi, \eta, \nu$ into (4.5) and (4.3) to obtain the physical quantities $\boldsymbol{v}$ and $\boldsymbol{B}$. Hence, the general barotropic magnetohydrodynamic problem is reduced from seven equations (2.1), (2.3), (2.4) and the additional constraint (2.2) to a problem of six first-order (in the temporal derivative) unconstrained equations. Moreover, the entire set of equations can be derived from the Lagrangian density $\hat{\mathscr{L}}$ which is what we were aiming to prove.

It should be emphasized that for non-trivial topologies it is necessary to assume that some of the variables introduced in the non-stationary formalism are non-single-valued. That is, it is necessary to introduce a number of branch cuts in order to define single-valued branches of the field variables. In turn, these cuts along with the six field variables constitute an extended number of dynamic variables. The number of necessary cuts depends on the flow and for complicated field topologies can lead to detailed book-keeping.

4.3. *The inverse problem*

In §4.2 we have shown that, given a set of functions $\alpha, \beta, \chi, \eta, \nu$ satisfying the set of equations described in the previous subsections, one can insert those functions into (4.5) and (4.3) to obtain the physical velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{B}$. In this subsection we will address the inverse problem; that is, suppose we are given the quantities $\boldsymbol{v}, \boldsymbol{B}$ and ρ, then how can one calculate the potentials $\alpha, \beta, \chi, \eta, \nu$? The treatment in this section will follow closely an analogous treatment for non-magnetic fluid dynamics given by Lynden-Bell & Katz (1981).

Consider a thin tube surrounding a magnetic field line as described in figure 1; the magnetic flux contained within the tube is

$$\Delta \Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \tag{4.16}$$

and the mass contained with the tube is

$$\Delta M = \int \rho \mathrm{d} \boldsymbol{l} \cdot \mathrm{d} \boldsymbol{S}, \tag{4.17}$$

in which $\mathrm{d}\boldsymbol{l}$ is a length element along the tube. Since the magnetic field lines move with the flow by virtue of (2.1) both the quantities $\Delta \Phi$ and ΔM are conserved and since the tube is thin we may define the conserved magnetic load:

$$\lambda = \frac{\Delta M}{\Delta \Phi} = \oint \frac{\rho}{B} \mathrm{d}l, \tag{4.18}$$

in which the above integral is performed along the field line. Obviously the parts of the line which go out of the flow to regions in which $\rho = 0$ have a null contribution to the integral. Notice that λ is a single-valued function that can be measured in principle. Since λ is conserved it satisfies the equation

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0. \tag{4.19}$$

By construction, surfaces of constant magnetic load move with the flow and contain magnetic field lines. Hence the gradient to such surfaces must be orthogonal to the field line:

$$\nabla \lambda \cdot \boldsymbol{B} = 0. \tag{4.20}$$

For a discussion of the possibility of surface- and volume-filling fields and the complications in the definition of λ then see § 6.3. Now consider an arbitrary comoving point on the magnetic field line and denote it by i, and consider an additional comoving point on the magnetic field line and denote it by r. The integral

$$\mu(r) = \int_i^r \frac{\rho}{B} \mathrm{d}l + \mu(i) \tag{4.21}$$

is also a conserved quantity which we may denote following Lynden-Bell & Katz (1981) as the magnetic metage. $\mu(i)$ is an arbitrary number which can be chosen differently for each magnetic line. By construction

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \tag{4.22}$$

Also it is easy to see that by differentiating along the magnetic field line we obtain

$$\nabla \mu \cdot \boldsymbol{B} = \rho. \tag{4.23}$$

Notice that μ will be generally a non-single-valued function; we will show later in this paper that symmetry to translations in μ will generate through the Noether theorem the conservation of the magnetic cross-helicity.

At this point we have two comoving coordinates of flow, namely λ, μ; obviously in a three-dimensional flow we also have a third coordinate. However, before defining the third coordinate we will find it useful to work not directly with λ but with a function of λ. Now consider the magnetic flux within a surface of constant load $\Phi(\lambda)$ as described in figure 2 (the figure was taken from Lynden-Bell & Katz 1981). The magnetic flux is a conserved quantity and depends only on the load λ of the surrounding surface. Now we define the quantity

$$\chi = \frac{\Phi(\lambda)}{2\pi}. \tag{4.24}$$

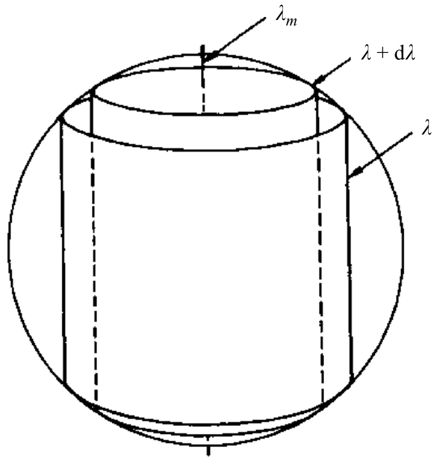

FIGURE 2. Surfaces of constant load (Lynden-Bell & Katz 1981).

Obviously χ satisfies the equations

$$\frac{d\chi}{dt} = 0, \qquad \boldsymbol{B} \cdot \nabla\chi = 0, \tag{4.25}$$

We will immediately show that this function is identical to Sakurai's function defined in (3.2). Let us now define an additional comoving coordinate η^*: since $\nabla\mu$ is not orthogonal to the $\boldsymbol{B}$ lines we can choose $\nabla\eta^*$ to be orthogonal to the $\boldsymbol{B}$ lines and not in the direction of the $\nabla\chi$ lines, that is we choose η^* not to depend only on χ. Since both $\nabla\eta^*$ and $\nabla\chi$ are orthogonal to $\boldsymbol{B}$, $\boldsymbol{B}$ must take the form

$$\boldsymbol{B} = A\nabla\chi \times \nabla\eta^*. \tag{4.26}$$

However, using (2.2) we have

$$\nabla \cdot \boldsymbol{B} = \nabla A \cdot (\nabla\chi \times \nabla\eta^*) = 0, \tag{4.27}$$

which implies that A is a function of χ, η^*. Now we can define a new comoving function η such that

$$\eta = \int_0^{\eta^*} A(\chi, \eta'^*)\, d\eta'^*, \qquad \frac{d\eta}{dt} = 0. \tag{4.28}$$

In terms of this function we recover the Sakurai presentation defined in (3.2):

$$\boldsymbol{B} = \nabla\chi \times \nabla\eta. \tag{4.29}$$

Hence we have shown how χ, η can be constructed for a known $\boldsymbol{B}, \rho$. Notice however, that η is defined in a non-unique way since one can redefine η for example by performing the following transformation: $\eta \rightarrow \eta + f(\chi)$ in which $f(\chi)$ is an arbitrary function. The comoving coordinates χ, η serve as labels of the magnetic field lines. Moreover the magnetic flux can be calculated as

$$\Phi = \int \boldsymbol{B} \cdot d\boldsymbol{S} = \int d\chi d\eta. \tag{4.30}$$

In the case that the surface integral is performed inside a load contour we obtain

$$\Phi(\lambda) = \int_\lambda d\chi\, d\eta = \chi \int_\lambda d\eta = \begin{cases} \chi[\eta] \\ \chi(\eta_{max} - \eta_{min}). \end{cases} \tag{4.31}$$

There are two cases involved; in one case the load surfaces are topological cylinders, and η is not single-valued and hence we obtain the upper value for $\Phi(\lambda)$. In a second case the load surfaces are topological spheres; in this case η is single-valued and has minimal η_{min} and maximal η_{max} values. Hence the lower value of $\Phi(\lambda)$ is obtained. For example in some cases η is identical to twice the latitude angle θ. In those cases $\eta_{min} = 0$ (value at the 'north pole') and $\eta_{max} = 2\pi$ (value at the 'south pole').

Comparing the above equation with (4.24) we derive that η can be either single-valued or not-single-valued and that its discontinuity across its cut in the non-single-valued case is $[\eta] = 2\pi$.

We will now show how the potentials α, β, ν can be derived. Let us calculate the vorticity $\boldsymbol{\omega}$ of the flow. By taking the curl of (4.5) we obtain

$$\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{v} = \nabla\alpha \times \nabla\chi + \nabla\beta \times \nabla\eta. \tag{4.32}$$

The following identities are derived:

$$\boldsymbol{\omega} \cdot \nabla\chi = (\nabla\beta \times \nabla\eta) \cdot \nabla\chi = -\nabla\beta \cdot \boldsymbol{B}, \tag{4.33}$$

$$\boldsymbol{\omega} \cdot \nabla\eta = (\nabla\alpha \times \nabla\chi) \cdot \nabla\eta = \nabla\alpha \cdot \boldsymbol{B}. \tag{4.34}$$

Now let us perform integrations along $\boldsymbol{B}$ lines starting from an arbitrary point denoted as i to another arbitrary point denoted as r.

$$\beta(r) = -\int_i^r \frac{\boldsymbol{\omega} \cdot \nabla\chi}{B}\, \mathrm{d}l + \beta(i), \tag{4.35}$$

$$\alpha(r) = \int_i^r \frac{\boldsymbol{\omega} \cdot \nabla\eta}{B}\, \mathrm{d}l + \alpha(i). \tag{4.36}$$

The numbers $\alpha(i), \beta(i)$ can be chosen in an arbitrary way for each magnetic field line. Hence we have derived (in a non-unique way) the values of the α, β functions. Finally we can use (4.5) to derive the function ν for any point s within the flow:

$$\nu(s) = \int_i^s (\boldsymbol{v} - \alpha\nabla\chi - \beta\nabla\eta) \cdot \mathrm{d}\boldsymbol{r} + \nu(i), \tag{4.37}$$

in which i is any arbitrary point within the flow. The result will not depend on the trajectory taken in the case that ν is single-valued. If ν is not single-valued one should introduce a cut which the integration trajectory should not cross.

4.4. *Stationary barotropic magnetohydrodynamics*

Stationary flows are a unique phenomenon of Eulerian fluid dynamics which have no counterpart in Lagrangian fluid dynamics. The stationary flow is defined by the fact that the physical fields $\boldsymbol{v}, \boldsymbol{B}, \rho$ do not depend on the temporal coordinate. This, however, does not imply that the corresponding potentials $\alpha, \beta, \chi, \eta, \nu$ are all functions of spatial coordinates alone. Moreover, it can be shown that choosing the potentials in such a way will lead to erroneous results in the sense that the stationary equations of motion cannot be derived from the Lagrangian density $\hat{\mathscr{L}}$ given in (4.15). However, this problem can be amended easily as follows. Let us choose α, β, χ, ν to depend on the spatial coordinates alone. Let us choose η such that

$$\eta = \bar{\eta} - t, \tag{4.38}$$

in which $\bar{\eta}$ is a function of the spatial coordinates. The Lagrangian density $\hat{\mathscr{L}}$ given in (4.15) will take the form

$$\hat{\mathscr{L}} = \rho(\beta - \varepsilon(\rho)) - \frac{1}{2}\rho(\nabla\nu + \alpha\nabla\chi + \beta\nabla\bar{\eta})^2 - \frac{1}{8\pi}(\nabla\chi \times \nabla\bar{\eta})^2. \tag{4.39}$$

The above functional can be compared with Vladimirov & Moffatt's (1995) equation (6.12) for incompressible flows in which their I is analogous to our β. Notice however, that while β is not a conserved quantity I is.

Varying the Lagrangian $\hat{L} = \int \hat{\mathscr{L}}\, d^3x$ with respect to $\nu, \alpha, \beta, \chi, \eta, \rho$ leads to the following equations:

$$\left.\begin{array}{lll} \nabla\cdot(\rho\hat{\boldsymbol{v}}) = 0, & \rho\hat{\boldsymbol{v}}\cdot\nabla\chi = 0, & \rho(\hat{\boldsymbol{v}}\cdot\nabla\bar{\eta} - 1) = 0, \\ \hat{\boldsymbol{v}}\cdot\nabla\alpha = \dfrac{\nabla\bar{\eta}\cdot\hat{\boldsymbol{J}}}{\rho}, & \hat{\boldsymbol{v}}\cdot\nabla\beta = -\dfrac{\nabla\chi\cdot\hat{\boldsymbol{J}}}{\rho}, & \beta = \tfrac{1}{2}\hat{\boldsymbol{v}}^2 + w. \end{array}\right\} \tag{4.40}$$

Calculations similar to those in previous subsections will show that these equations lead to the stationary barotropic magnetohydrodynamic equations:

$$\nabla\times(\hat{\boldsymbol{v}}\times\hat{\boldsymbol{B}}) = 0, \tag{4.41}$$

$$\rho(\hat{\boldsymbol{v}}\cdot\nabla)\hat{\boldsymbol{v}} = -\nabla p(\rho) + \frac{(\nabla\times\hat{\boldsymbol{B}})\times\hat{\boldsymbol{B}}}{4\pi}. \tag{4.42}$$

5. The simplified Hamiltonian formalism

Let us derive the conjugate momenta of the variables appearing in the Lagrangian density $\hat{\mathscr{L}}$ defined in (4.15). A simple calculation will yield

$$\pi_\nu \equiv \frac{\partial\hat{\mathscr{L}}}{\partial(\partial\nu/\partial t)} = -\rho, \quad \pi_\chi \equiv \frac{\partial\hat{\mathscr{L}}}{\partial(\partial\chi/\partial t)} = -\rho\alpha, \quad \pi_\eta \equiv \frac{\partial\hat{\mathscr{L}}}{\partial(\partial\eta/\partial t)} = -\rho\beta. \tag{5.1}$$

The rest of the canonical momenta $\pi_\rho, \pi_\alpha, \pi_\beta$ are null. It thus seems that the six functions appearing in the Lagrangian density $\hat{\mathscr{L}}$ can be divided into 'approximate' conjugate pairs: $(\nu,\rho), (\chi,\alpha), (\eta,\beta)$. The Hamiltonian density $\hat{\mathscr{H}}$ can be now calculated as follows:

$$\hat{\mathscr{H}} = \pi_\nu\frac{\partial\nu}{\partial t} + \pi_\chi\frac{\partial\chi}{\partial t} + \pi_\eta\frac{\partial\eta}{\partial t} - \hat{\mathscr{L}} = \rho\left[\varepsilon(\rho) + \tfrac{1}{2}\hat{\boldsymbol{v}}^2\right] + \frac{1}{8\pi}\hat{\boldsymbol{B}}^2, \tag{5.2}$$

in which $\hat{\boldsymbol{v}}$ is defined in (4.5) and $\hat{\boldsymbol{B}}$ is defined in (4.3). This Hamiltonian was previously introduced by Frenkel *et al.* (1982) using somewhat different variables ($\lambda = \chi, \Lambda = \eta, \mu = \pi_\chi, M = \pi_\eta, \phi = \nu$). The equations derived from the above Hamiltonian density are similar to (4.2), (4.7) and (4.10) and will not be re-derived here. While Frenkel *et al.* (1982) have postulated the Hamiltonian density appearing in (5.2), this Hamiltonian is here derived from a Lagrangian.

6. Simplified variational principle of stationary barotropic magnetohydrodynamics

In the previous section we have shown that barotropic magnetohydrodynamics can be described in terms of six first-order differential equations or of an action principle from which those equations can be derived. This formalism was shown to apply to both stationary and non-stationary magnetohydrodynamics. Although for non-stationary magnetohydrodynamics we do not know at present how the number of functions can be further reduced, for stationary barotropic magnetohydrodynamics the situation is quite different. We will show that for stationary barotropic magnetohydrodynamics three functions will suffice.

Consider (4.25), for a stationary flow it takes the form

$$\boldsymbol{v}\cdot\nabla\chi = 0. \tag{6.1}$$

Hence $\boldsymbol{v}$ can take the form

$$\boldsymbol{v} = \frac{\nabla\chi \times \boldsymbol{K}}{\rho}. \tag{6.2}$$

However, the velocity field must satisfy the stationary mass conservation equation (2.3):

$$\nabla\cdot(\rho\boldsymbol{v}) = 0. \tag{6.3}$$

We see that a sufficient condition (although not necessary) for $\boldsymbol{v}$ to solve (6.3) is that $\boldsymbol{K}$ takes the form $\boldsymbol{K} = \nabla N$, where N is an arbitrary function. Thus, $\boldsymbol{v}$ may take the form

$$\boldsymbol{v} = \frac{\nabla\chi \times \nabla N}{\rho}. \tag{6.4}$$

Let us now calculate $\boldsymbol{v}\times\boldsymbol{B}$ in which $\boldsymbol{B}$ is given by Sakurai's presentation (4.3):

$$\begin{aligned}\boldsymbol{v}\times\boldsymbol{B} &= \left(\frac{\nabla\chi \times \nabla N}{\rho}\right)\times(\nabla\chi\times\nabla\eta)\\ &= \frac{1}{\rho}\nabla\chi(\nabla\chi\times\nabla N)\cdot\nabla\eta.\end{aligned} \tag{6.5}$$

Since the flow is stationary N can be at most a function of the three comoving coordinates $\chi, \mu, \bar{\eta}$ defined in §§4.3 and 4.4, hence

$$\nabla N = \frac{\partial N}{\partial\chi}\nabla\chi + \frac{\partial N}{\partial\mu}\nabla\mu + \frac{\partial N}{\partial\bar{\eta}}\nabla\bar{\eta}. \tag{6.6}$$

Inserting (6.6) into (6.5) will yield

$$\boldsymbol{v}\times\boldsymbol{B} = \frac{1}{\rho}\nabla\chi\frac{\partial N}{\partial\mu}(\nabla\chi\times\nabla\mu)\cdot\nabla\bar{\eta}. \tag{6.7}$$

Rearranging terms and using Sakurai's presentation (4.3) we can simplify the above equation and obtain

$$\boldsymbol{v}\times\boldsymbol{B} = -\frac{1}{\rho}\nabla\chi\frac{\partial N}{\partial\mu}(\nabla\mu\cdot\boldsymbol{B}). \tag{6.8}$$

However, using (4.23) this will simplify to the form

$$\boldsymbol{v}\times\boldsymbol{B} = -\nabla\chi\frac{\partial N}{\partial\mu}. \tag{6.9}$$

Now let us consider (2.1); for stationary flows this will take the form

$$\nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = 0. \tag{6.10}$$

Inserting (6.8) into (4.41) will lead to the equation

$$\nabla\left(\frac{\partial N}{\partial\mu}\right)\times\nabla\chi = 0. \tag{6.11}$$

However, since N is at most a function of $\chi, \mu, \bar{\eta}$ it follows that $\partial N/\partial\mu$ is some function of χ:

$$\frac{\partial N}{\partial\mu} = -F(\chi). \tag{6.12}$$

This can be easily integrated to yield

$$N = -\mu F(\chi) + G(\chi, \bar{\eta}). \tag{6.13}$$

Inserting this back into (6.4) will yield

$$\boldsymbol{v} = \frac{\nabla\chi \times (-F(\chi)\nabla\mu + (\partial G/\partial\bar{\eta})\nabla\bar{\eta})}{\rho}. \tag{6.14}$$

Let us now replace the set of variables $\chi, \bar{\eta}$ with a new set $\chi', \bar{\eta}'$ such that

$$\chi' = \int F(\chi)\,\mathrm{d}\chi, \qquad \bar{\eta}' = \frac{\bar{\eta}}{F(\chi)}. \tag{6.15}$$

This will not have any effect on the Sakurai representation given in (4.3) since

$$\boldsymbol{B} = \nabla\chi \times \nabla\eta = \nabla\chi \times \nabla\bar{\eta} = \nabla\chi' \times \nabla\bar{\eta}'. \tag{6.16}$$

However, the velocity will have a simpler representation and will take the form

$$\boldsymbol{v} = \frac{\nabla\chi' \times \nabla(-\mu + G'(\chi', \bar{\eta}'))}{\rho}, \tag{6.17}$$

in which $G' = G/F$. At this point one should remember that μ was defined in (4.21) up to an arbitrary constant which can vary between magnetic field lines. Since the lines are labelled by their $\chi', \bar{\eta}'$ values it follows that we can add an arbitrary function of $\chi', \bar{\eta}'$ to μ without affecting its properties. Hence we can define a new μ' such that

$$\mu' = \mu - G'(\chi', \bar{\eta}'). \tag{6.18}$$

Notice that μ' can be multi-valued; this will be discussed in more detail in § 6.3. Inserting (6.18) into (6.17) will lead to a simplified equation for $\boldsymbol{v}$:

$$\boldsymbol{v} = \frac{\nabla\mu' \times \nabla\chi'}{\rho}. \tag{6.19}$$

In the following the primes on $\chi, \mu, \bar{\eta}$ will be ignored. The above equation is analogous to Vladimirov & Moffatt's (1995) equation (7.11) for incompressible flows, in which our μ and χ play the part of their A and Ψ. It is obvious that $\boldsymbol{v}$ satisfies the following set of equations:

$$\boldsymbol{v}\cdot\nabla\mu = 0, \qquad \boldsymbol{v}\cdot\nabla\chi = 0, \qquad \boldsymbol{v}\cdot\nabla\bar{\eta} = 1. \tag{6.20}$$

To derive the right-hand equation we have used both (4.22) and (4.3). Hence μ, χ are both comoving and stationary. As for $\bar{\eta}$ it satisfies the same equation as $\bar{\eta}$ defined in (4.38) as can be seen from (4.40). It can be easily seen that if

$$basis = (\nabla\chi, \nabla\bar{\eta}, \nabla\mu) \tag{6.21}$$

is a local vector basis at any point in space then their exists a dual basis:

$$dual\ \ basis = \frac{1}{\rho}(\nabla\bar{\eta} \times \nabla\mu, \nabla\mu \times \nabla\chi, \nabla\chi \times \nabla\bar{\eta}) = \left(\frac{\nabla\bar{\eta} \times \nabla\mu}{\rho}, \boldsymbol{v}, \frac{\boldsymbol{B}}{\rho}\right) \tag{6.22}$$

such that

$$basis_i \cdot dual\ \ basis_j = \delta_{ij}, \qquad i, j \in [1, 2, 3], \tag{6.23}$$

in which δ_j is Kronecker's delta. Hence while the surfaces $\chi, \mu, \bar{\eta}$ generate a local vector basis for space, the physical fields of interest $\boldsymbol{v}, \boldsymbol{B}$ are part of the dual basis. By vector multiplying $\boldsymbol{v}$ and $\boldsymbol{B}$ and using (6.19), (4.3) we obtain

$$\boldsymbol{v} \times \boldsymbol{B} = \nabla\chi; \tag{6.24}$$

this means that both $\boldsymbol{v}$ and $\boldsymbol{B}$ lie on χ surfaces and provide a vector basis for this two-dimensional surface. The above equation can be compared with Vladimirov & Moffatt (1995) equation (5.6) for incompressible flows in which their J is analogous to our χ.

6.1. *The action principle*

In the first part of this section we have shown that if the velocity field $\boldsymbol{v}$ is given by (6.19) and the magnetic field $\boldsymbol{B}$ is given by the Sakurai representation (4.3) then (2.1), (2.2), (2.3) are satisfied automatically for stationary flows. To complete the set of equations we will show how the Euler (2.4) can be derived from the action given in (2.6) in which both $\boldsymbol{v}$ and $\boldsymbol{B}$ are given by (6.19) and (4.3) respectively and the density ρ is given by (4.22):

$$\rho = \nabla\mu\cdot\boldsymbol{B} = \nabla\mu\cdot(\nabla\chi\times\nabla\eta) = \frac{\partial(\chi,\eta,\mu)}{\partial(x,y,z)}. \tag{6.25}$$

In this case the Lagrangian density of (2.6) will take the form

$$\mathscr{L} = \rho\left(\frac{1}{2}\left(\frac{\nabla\mu\times\nabla\chi}{\rho}\right)^2 - \varepsilon(\rho)\right) - \frac{(\nabla\chi\times\nabla\eta)^2}{8\pi} \tag{6.26}$$

and can be seen explicitly to depend on only three functions. Let us make arbitrary small variations $\delta\alpha_i = (\delta\chi, \delta\eta, \delta\mu)$ of the functions $\alpha_i = (\chi, \eta, \mu)$. Let us define a Δ variation that does not modify the α_i, such that

$$\Delta\alpha_i = \delta\alpha_i + (\boldsymbol{\xi}\cdot\nabla)\alpha_i = 0, \tag{6.27}$$

in which $\boldsymbol{\xi}$ is the Lagrangian displacement, thus

$$\delta\alpha_i = -\nabla\alpha_i\cdot\boldsymbol{\xi}, \tag{6.28}$$

which will lead to

$$\boldsymbol{\xi} \equiv -\frac{\partial\boldsymbol{r}}{\partial\alpha_i}\delta\alpha_i. \tag{6.29}$$

Making a variation of ρ given in (6.25) with respect to α_i will yield (2.8). Furthermore, taking the variation of $\boldsymbol{B}$ given by Sakurai's representation (4.3) with respect to α_i will yield (2.9). It remains to calculate $\delta\boldsymbol{v}$ by varying (6.19); this will yield

$$\delta\boldsymbol{v} = -\frac{\delta\rho}{\rho}\boldsymbol{v} + \frac{1}{\rho}\nabla\times(\rho\boldsymbol{\xi}\times\boldsymbol{v}). \tag{6.30}$$

Inserting (2.8), (2.9), (6.30) into (2.7) will yield

$$\begin{aligned}\delta\mathscr{L} &= \boldsymbol{v}\cdot\nabla\times(\rho\boldsymbol{\xi}\times\boldsymbol{v}) - \frac{\boldsymbol{B}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})}{4\pi} - \delta\rho\left(\tfrac{1}{2}\boldsymbol{v}^2 + w\right)\\ &= \boldsymbol{v}\cdot\nabla\times(\rho\boldsymbol{\xi}\times\boldsymbol{v}) - \frac{\boldsymbol{B}\cdot\nabla\times(\boldsymbol{\xi}\times\boldsymbol{B})}{4\pi} + \nabla\cdot(\rho\boldsymbol{\xi})\left(\tfrac{1}{2}\boldsymbol{v}^2 + w\right).\end{aligned} \tag{6.31}$$

Using the well-known vector identity

$$\boldsymbol{A}\cdot\nabla\times(\boldsymbol{C}\times\boldsymbol{A}) = \nabla\cdot((\boldsymbol{C}\times\boldsymbol{A})\times\boldsymbol{A}) + (\boldsymbol{C}\times\boldsymbol{A})\cdot\nabla\times\boldsymbol{A} \tag{6.32}$$

and the theorem of Gauss we can now write (2.7) in the form

$$\begin{aligned}\delta A = \int \mathrm{d}t\Bigg\{ \oint \mathrm{d}\boldsymbol{S}\cdot\left[\rho(\boldsymbol{\xi}\times\boldsymbol{v})\times\boldsymbol{v} - \frac{(\boldsymbol{\xi}\times\boldsymbol{B})\times\boldsymbol{B}}{4\pi} + \left(\tfrac{1}{2}\boldsymbol{v}^2 + w\right)\rho\boldsymbol{\xi}\right]\\ + \int \mathrm{d}^3x\,\boldsymbol{\xi}\cdot\left[\rho\boldsymbol{v}\times\boldsymbol{\omega} + \boldsymbol{J}\times\boldsymbol{B} - \rho\nabla\left(\tfrac{1}{2}\boldsymbol{v}^2 + w\right)\right]\Bigg\}.\end{aligned} \tag{6.33}$$

The time integration is of course redundant in the above expression. Also notice that we have used the current definition (2.5) and the vorticity definition (4.32). Suppose now that $\delta A = 0$ for a $\boldsymbol{\xi}$ such that the boundary term (including both the boundary of the domain and relevant cuts) in the above equation is null but that $\boldsymbol{\xi}$ is otherwise arbitrary; this entails the equation

$$\rho \boldsymbol{v} \times \boldsymbol{\omega} + \boldsymbol{J} \times \boldsymbol{B} - \rho \nabla\left(\tfrac{1}{2}\boldsymbol{v}^2 + w\right) = 0. \tag{6.34}$$

Using the well-known vector identity

$$\tfrac{1}{2}\nabla(\boldsymbol{v}^2) = (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \tag{6.35}$$

and rearranging terms we recover the stationary Euler equation

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \boldsymbol{J} \times \boldsymbol{B}. \tag{6.36}$$

6.2. *The case of an axisymmetric magnetic field*

Consider an axisymmetric magnetic field such that the magnetic field is dependent only on the coordinate R, which is the distance from the axis of symmetry, and the coordinate z, which is the distance along the axis of symmetry from an arbitrary origin on the axis. Thus

$$\boldsymbol{B} = \boldsymbol{B}(R, z). \tag{6.37}$$

Any axisymmetric magnetic field satisfying (2.2) can be represented in the form

$$\boldsymbol{B} = \nabla P \times \nabla\left(\frac{\phi}{2\pi}\right) + 2\pi R B_\phi \nabla\left(\frac{\phi}{2\pi}\right), \tag{6.38}$$

in which ϕ is the azimuthal angle defined in the conventional way and B_ϕ is the component of $\boldsymbol{B}$ in the ϕ-direction. The function $P = P(R, z)$ is the flux through a circle of radius R at height z:

$$P(R, z) = \int_{(R,z)} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 2\pi \int_0^R B_z(R', z) R' \, \mathrm{d}R'. \tag{6.39}$$

For finite field configurations P will have a maximum $P_m = P(R_m, z_m)$ at some R_m, z_m. This circle $R = R_m$ will form a line toroid with the other constant-P surfaces nearby forming a nested set. There can be several such local maxima with local nested toroids in a general configuration but the simpler case has just one.

Let us study the relations between the functions P, B_ϕ and the functions χ, η given in (3.2). Assuming that the density ρ is axisymmetric one can see the magnetic load defined in (4.18) is also axisymmetric and that the surfaces of constant load are surfaces of revolution around the axis of symmetry. From (4.24) we deduce that $\chi = \chi(R, z)$. Expressing (3.2) in terms of the coordinates R, ϕ, z results in

$$\boldsymbol{B} = \left(-\frac{1}{\boldsymbol{R}}\partial_z \chi \partial_\phi \eta\right)\hat{\boldsymbol{R}} + (\partial_z \chi \partial_{\boldsymbol{R}} \eta - \partial_{\boldsymbol{R}} \chi \partial_z \eta)\hat{\boldsymbol{\phi}} + \left(\frac{1}{\boldsymbol{R}}\partial_{\boldsymbol{R}} \chi \partial_\phi \eta\right)\hat{\boldsymbol{z}}, \tag{6.40}$$

in which ∂_y is a shorthand notation for $\partial/\partial y$ and $\hat{\boldsymbol{y}}$ is a unit vector perpendicular to the constant-y surface. Comparing (6.40) with (6.38) we arrive at the set of equations

$$\partial_z P = \partial_z \chi \partial_\phi(2\pi\eta), \qquad \partial_R P = \partial_R \chi \partial_\phi(2\pi\eta), \tag{6.41}$$

from which we derive the equation

$$\partial_z P \partial_R \chi - \partial_R P \partial_z \chi = 0 \Rightarrow \nabla P \times \nabla \chi = 0. \tag{6.42}$$

Hence

$$P = F(\chi), \tag{6.43}$$

where F is an arbitrary function. We deduce that P is just another type of labelling of the load surfaces. Thus (6.41) will lead to

$$\partial_\phi(2\pi\eta) = \frac{dP}{d\chi} \Rightarrow \eta = \frac{\phi}{2\pi}\frac{dP}{d\chi} + \tilde{\eta}(R, z). \tag{6.44}$$

This should be compared with the result of Yang *et al.* (1986). Substituting the above result in (6.40) will lead to the equation

$$B_\phi \hat{\boldsymbol{\phi}} = (\partial_z \chi \partial_R \tilde{\eta} - \partial_R \chi \partial_z \tilde{\eta})\hat{\boldsymbol{\phi}} = \nabla\chi \times \nabla\tilde{\eta}. \tag{6.45}$$

This can also be written as

$$B_\phi = \hat{\boldsymbol{\phi}} \cdot (\nabla\chi \times \nabla\tilde{\eta}) = (\hat{\boldsymbol{\phi}} \times \nabla\chi) \cdot \nabla\tilde{\eta}. \tag{6.46}$$

Hence B_ϕ is proportional to the gradient of $\tilde{\eta}$ along the $\hat{\boldsymbol{\phi}} \times \nabla\chi$ direction. Since $\hat{\boldsymbol{\phi}} \times \nabla\chi$ is known we can integrate along this vector to obtain a non-unique solution for $\tilde{\eta}$:

$$\tilde{\eta} = \int \frac{B_\phi}{|\hat{\boldsymbol{\phi}} \times \nabla\chi|}\, dl, \tag{6.47}$$

in which dl is a line element along the $\hat{\boldsymbol{\phi}} \times \nabla\chi$ line.

6.3. *The case of a magnetic field on a toroid*

Our previous definition of the surfaces of constant load given in (4.18) is ambiguous when the field lines are 'surface filling' e.g. on a toroid, and give no result when the field lines are 'volume filling'. At equilibrium $\boldsymbol{B}$ and $\boldsymbol{v}$ lie in surfaces (there is an exception when $\boldsymbol{B}$ and $\boldsymbol{v}$ are parallel and fill volumes). Our former considerations apply unchanged if these surfaces have the topology of cylinders but they need generalization when the surfaces have the topology of toroids nested on a line (a similar discussion in which non-magnetic fluids are considered can be found in Lynden-Bell 1996). We consider a surface Σ spanning that line toroid. Each toroid T will meet Σ in a loop. Consider the magnetic flux $\Phi(T)$ through that part of Σ within the loop and the mass enclosed by the toroid $m(T)$. Then the mass outside the toroid is $\tilde{m}(T) = M - m(T)$. Now express $\tilde{m}$ as a function $\tilde{m}(\Phi)$ of the magnetic flux Φ, then a definition of magnetic load analogous to that for 'cylinders' parallel to the axis is

$$\lambda = \frac{d\tilde{m}}{d\Phi} \tag{6.48}$$

However, there are now two loads corresponding to the two fluxes associated with a given toroid. The other load is obtained by taking a cut across the 'short' circle section of the torus, say of constant ϕ. The magnetic flux Φ^* through such a cross-section may be expressed as a function of the total mass $m(T)$ within the toroid and

$$\lambda^* = \frac{dm}{d\Phi^*} \tag{6.49}$$

is a second different load. Of course it is also permissible to re-express the flux Φ^* as a function of the flux Φ; then we find

$$\lambda^* = \frac{dm}{d\Phi^*} = \frac{dm/d\Phi}{d\Phi^*/d\Phi} = -\frac{\lambda}{d\Phi^*/d\Phi}. \tag{6.50}$$

The surfaces of constant λ are of course the toroids T which also have λ^* constant.

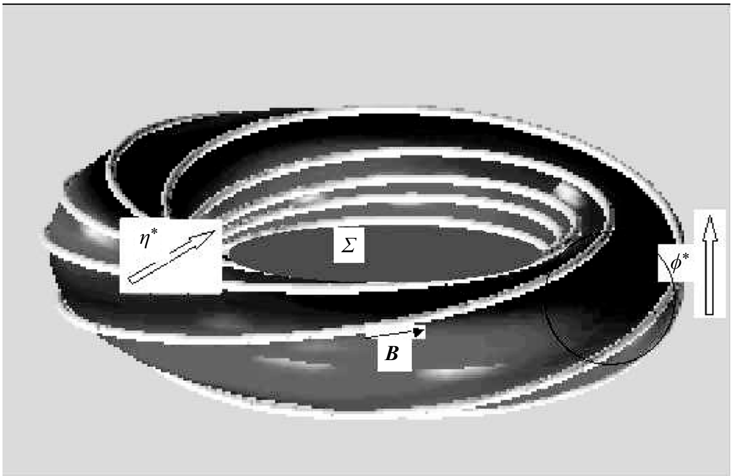

FIGURE 3. A torus of magnetic field lines.

A similar problem may arise with the definition of the magnetic metage defined in (4.21). We may wish to define this quantity using the magnetic field $\boldsymbol{B}$ and velocity field $\boldsymbol{v}$. Since those vectors provide a vector basis on the load surface, they can be combined in such a way, say $\boldsymbol{B} + \gamma(\chi)\boldsymbol{v}$, to create a vector which is directed along the large loop of the toroid. (A different γ will leave only twists around the short way.) This combination represents an unwinding of the field lines so that they no longer twist around the short (long) way. Those loops can be thought as composing the surface Σ. Another surface Σ' also composed of such untwisted loops can be so chosen that the mass between Σ and Σ' and between loads λ and $\lambda + \mathrm{d}\lambda$ is some fixed fraction of $(\mathrm{d}m/\mathrm{d}\lambda)\,\mathrm{d}\lambda$. Such Σ' form suitable constant-metage surfaces μ corresponding to partial loads λ. Notice that $2\pi\mu$ then describes the angle from Σ turned around the toroid by the short way to reach any chosen point. Similar use of the other load λ^* allows us to define another generalized angle μ^* measured around the long way. A somewhat less physical approach is given below.

Let us consider a toroid of constant magnetic load. Dungey (1958) has considered the case in which magnetic field lines lie on a torus. He has shown that one of the functions (i.e. η) involved in the representation (3.2) should be non-single-valued and therefore a cut should be introduced.

In order to obtain a simple looking cut we will replace the previous set of functions μ, η with a new set ϕ^*, η^*, which will be defined as follows:

$$\phi^* \equiv \frac{\mu + G(\chi)\eta}{\Omega(\chi)}, \qquad \eta^* \equiv \frac{\eta - \phi^*}{f(\chi)}, \tag{6.51}$$

where G, Ω, f are arbitrary functions of χ. Therefore μ and η can be given in terms of ϕ^*, η^* as

$$\eta = \phi^* + f(\chi)\eta^*, \qquad \mu = -G(\chi)\eta + \Omega(\chi)\phi^*; \tag{6.52}$$

η^* can be considered as an angle varying over the small circle of the torus, while ϕ^* can be considered as an angle varying over the large circle of the torus as in figure 3. On the torus of constant magnetic load the ϕ^*, η^* functions have a simple 'cut' structure. The above equation can also serve as a 'definition' of μ. Inserting

(6.52) into (3.2) and (6.19) will result in the following set of equations:

$$\left.\begin{aligned} \boldsymbol{B} &= \nabla\chi \times \nabla\phi^* + f(\chi)\nabla\chi \times \nabla\eta^*, \\ \boldsymbol{v} &= G(\chi)\tfrac{\boldsymbol{B}}{\rho} + \Omega(\chi)\tfrac{\nabla\phi^* \times \nabla\chi}{\rho}. \end{aligned}\right\} \tag{6.53}$$

Hence $\boldsymbol{B}$ is partitioned into two vectors circulating along the small and large circles of the torus, while $\boldsymbol{v}$ has two vector components, one along the magnetic field $\boldsymbol{B}$ and another along the small circle.

7. Topological constants of motion

Magnetohydrodynamics is known to have the following two topological constants of motion: one is the magnetic helicity

$$\mathscr{H}_M \equiv \int \boldsymbol{B} \cdot \boldsymbol{A}\, \mathrm{d}^3x, \tag{7.1}$$

which is known to measure the degree of knottiness of lines of the magnetic field $\boldsymbol{B}$ Moffatt (1969). The domain of integration in (7.1) is the entire space; obviously regions containing a null magnetic field will have a null contribution to the integral. In the above equation $\boldsymbol{A}$ is the vector potential defined implicitly by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}. \tag{7.2}$$

The other topological constant is the magnetic cross-helicity:

$$\mathscr{H}_C \equiv \int \boldsymbol{B} \cdot \boldsymbol{v}\, \mathrm{d}^3x, \tag{7.3}$$

characterizing the degree of cross-knottiness of the magnetic field and velocity lines. The domain of integration in (7.3) is the magnetohydrodynamic flow domain.

7.1. *Representation in terms of the magnetohydrodynamic potentials*

Let us write the topological constants given in (7.1) and (7.3) in terms of the magnetohydrodynamic potentials α, β, χ, η, ν, ρ introduced in previous sections.

First let us combine (3.2) with (7.2) to obtain the equation

$$\nabla \times (\boldsymbol{A} - \chi\nabla\eta) = 0; \tag{7.4}$$

this leads immediately to the result

$$\boldsymbol{A} = \chi\nabla\eta + \nabla\zeta, \tag{7.5}$$

in which ζ is some function. Let us now calculate the scalar product $\boldsymbol{B} \cdot \boldsymbol{A}$:

$$\boldsymbol{B} \cdot \boldsymbol{A} = (\nabla\chi \times \nabla\eta) \cdot \nabla\zeta. \tag{7.6}$$

However, since we have the local vector basis $(\nabla\chi, \nabla\eta, \nabla\mu)$ we can write $\nabla\zeta$ as

$$\nabla\zeta = \frac{\partial\zeta}{\partial\chi}\nabla\chi + \frac{\partial\zeta}{\partial\mu}\nabla\mu + \frac{\partial\zeta}{\partial\eta}\nabla\eta. \tag{7.7}$$

Hence we can write

$$\boldsymbol{B} \cdot \boldsymbol{A} = \frac{\partial\zeta}{\partial\mu}(\nabla\chi \times \nabla\eta) \cdot \nabla\mu = \frac{\partial\zeta}{\partial\mu}\frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{7.8}$$

Let us think of the entire space outside the magnetohydrodynamic domain as containing low-density matter; in this case we can define the metage μ over the entire portion of space containing magnetic field lines and the integration domain of (7.1) and (7.3) coincide. Now we can insert (7.8) into (7.1) to obtain the expression

$$\mathscr{H}_M = \int \frac{\partial \zeta}{\partial \mu} \, d\mu \, d\chi \, d\eta. \tag{7.9}$$

Note that in some scenarios it may be that the flow domain should be divided into patches in which different definitions of μ, χ, η apply to different domains; we do not see this as a limitation for our formalism since the topology of the flow is conserved by the equations of magnetohydrodynamics. In those cases $\mathscr{H}_M$ should be calculated as the sum of the contributions from each patch. We can think of the magnetohydrodynamic domain as composed of thin closed tubes of magnetic lines each labelled by (χ, η). Performing the integration along such a thin tube in the metage direction results in

$$\oint_{\chi,\eta} \frac{\partial \zeta}{\partial \mu} \, d\mu = [\zeta]_{\chi,\eta}, \tag{7.10}$$

in which $[\zeta]_{\chi,\eta}$ is the discontinuity of the function ζ along its cut. Thus a thin tube of magnetic lines in which ζ is single-valued does not contribute to the magnetic helicity integral. Inserting (7.10) into (7.9) will result in

$$\mathscr{H}_M = \int [\zeta]_{\chi,\eta} \, d\chi \, d\eta = \int [\zeta] \, d\Phi, \tag{7.11}$$

in which we have used (4.30). Hence

$$[\zeta] = \frac{d\mathscr{H}_M}{d\Phi}, \tag{7.12}$$

and the discontinuity of ζ is thus the density of magnetic helicity per unit of magnetic flux in a tube. We deduce that the Sakurai representation does not entail zero magnetic helicity, rather it is perfectly consistent with non-zero magnetic helicity as was demonstrated above and in agreement to the claims made by Frenkel *et al.* (1982). Notice however, that the topological structure of the magnetohydrodynamic flow constrains the gauge freedom which is usually attributed to vector potential $\boldsymbol{A}$ and limits it to single-valued functions. Moreover, while the choice of $\boldsymbol{A}$ is arbitrary since one can add to $\boldsymbol{A}$ an arbitrary gradient of a single-valued function which may lead to different choices of ζ, the discontinuity value $[\zeta]$ is not arbitrary and has a physical meaning given above.

Let us now introduce the velocity expression (4.5) and calculate the scalar product of $\boldsymbol{B}$ and $\boldsymbol{v}$; using the same arguments as in the previous paragraph will lead to the expression

$$\boldsymbol{v} \cdot \boldsymbol{B} = \frac{\partial \nu}{\partial \mu} (\nabla \chi \times \nabla \eta) \cdot \nabla \mu = \frac{\partial \nu}{\partial \mu} \frac{\partial(\chi, \eta, \mu)}{\partial(x, y, z)}. \tag{7.13}$$

Inserting (7.13) into (7.3) will result in

$$\mathscr{H}_C = \int \frac{\partial \nu}{\partial \mu} \, d\mu \, d\chi \, d\eta. \tag{7.14}$$

We can think of the magnetohydrodynamic domain as composed of thin closed tubes of magnetic lines each labelled by (χ, η). Performing the integration along such a thin tube in the metage direction results in

$$\oint_{\chi,\eta} \frac{\partial \nu}{\partial \mu} \, d\mu = [\nu]_{\chi,\eta}, \tag{7.15}$$

in which $[\nu]_{\chi,\eta}$ is the discontinuity of the function ν along its cut. Thus a thin tube of magnetic lines in which ν is single-valued does not contribute to the cross-helicity integral. Inserting (7.15) into (7.14) will result in

$$\mathscr{H}_C = \int [\nu]_{\chi,\eta}\,\mathrm{d}\chi\,\mathrm{d}\eta = \int [\nu]\,\mathrm{d}\Phi, \tag{7.16}$$

in which we have used (4.30). Hence

$$[\nu] = \frac{\mathrm{d}\mathscr{H}_C}{\mathrm{d}\Phi}, \tag{7.17}$$

and the discontinuity of ν is thus the density of cross-helicity per unit of magnetic flux. We deduce that a flow with null cross-helicity will have a single-valued ν function or alternatively a non-single-valued ν will entail a non-zero cross-helicity. Furthermore, from (4.7) it is obvious that

$$\frac{\mathrm{d}[\nu]}{\mathrm{d}t} = 0. \tag{7.18}$$

We conclude that not only is the magnetic cross-helicity conserved as an integral quantity of the entire magnetohydrodynamic domain but also the (local) density of cross-helicity per unit of magnetic flux is a conserved quantity as well.

In the following subsections we give simple examples which will demonstrate some of the general assertions of this paragraph.

7.2. *A helical stratified magnetic field*

Consider a magnetohydrodynamic flow of uniform density ρ. Furthermore assume that the flow contains a helical stratified magnetic field:

$$\boldsymbol{B} = \begin{cases} 2B_\perp(1 - R/a)\hat{\boldsymbol{\phi}} + B_{z0}\hat{\boldsymbol{z}}, & R < a, \\ 0, & R > a, \end{cases} \tag{7.19}$$

in which R, $\boldsymbol{\phi}$, z are the standard cylindrical coordinates, $\hat{R}$, $\hat{\boldsymbol{\phi}}$, $\hat{z}$ are the corresponding unit vectors and B_{z0}, $B_\perp$ are constants. The magnetic field is contained in a cylinder of Radius a and is independent of z. A possible choice of the vector potential $\boldsymbol{A}$ is:

$$\boldsymbol{A} = \begin{cases} B_{z0}x\,\hat{\boldsymbol{y}} + B_\perp a(1 - R/a)^2\hat{\boldsymbol{z}}, & R < a, \\ 0, & R > a, \end{cases} \tag{7.20}$$

in which $\hat{\boldsymbol{y}}$ is a unit vector in the y-direction. Let us calculate the magnetic helicity of the field using (7.1). In order to obtain a finite magnetic helicity we assume that the field is contained between the planes $z = 0$ and $z = 1$; furthermore we assume that the planes $z = 0$ and $z = 1$ can be identified such that the magnetic field lines are closed. Thus the domain becomes a topological torus. Inserting (7.19) and (7.20) into (7.1) will result in

$$\mathscr{H}_M \equiv \int \boldsymbol{B}\cdot\boldsymbol{A}\,\mathrm{d}^3x = \frac{\pi}{3}a^3 B_{z0}B_\perp. \tag{7.21}$$

First let us calculate to load using (4.18) (we assume that $R < a$ in the following calculations); we obtain

$$\lambda = \rho\frac{4\pi B_\perp(1 - R/a)R + B_{z0}}{B_{z0}^2 + 4B_\perp^2(1 - R/a)^2} = \lambda(R). \tag{7.22}$$

hence the load surfaces are cylinders. The χ function can now be calculated according to (4.24) to yield the value

$$\chi = \tfrac{1}{2} B_{z0} R^2. \tag{7.23}$$

Solving (4.29) for η we obtain the following non-unique solution:

$$\eta = \frac{2B_\perp}{B_{z0}} \left(1 - \frac{R}{a}\right) \frac{z}{a} + \phi. \tag{7.24}$$

Substituting (7.23), (7.24) and (7.20) into (7.5) we can solve for ζ and obtain

$$\zeta = B_\perp z(a - R) + \tfrac{1}{2} B_{z0} xy. \tag{7.25}$$

Since we have identified the $z = 0$ and $z = 1$ planes the z coordinate is not single-valued and therefore ζ is a non-single-valued function which has a discontinuity value:

$$[\zeta] = B_\perp (a - R). \tag{7.26}$$

Thus we can calculate the magnetic helicity using (7.11) and obtain

$$\mathscr{H}_M = \int [\zeta]\, \mathrm{d}\Phi = \frac{\pi}{3} a^3 B_{z0} B_\perp, \tag{7.27}$$

which coincides with the result of (7.9).

7.3. *Self-knotted magnetic field lines on nested tori*

Consider a magnetohydrodynamic flow of uniform density ρ. Furthermore assume (following Moffatt 1969) that the flow contains a vector potential:

$$\boldsymbol{A} = \nabla\Psi \times \nabla\phi + \alpha\Psi\nabla\phi = \frac{1}{R}\partial_R \Psi \hat{z} - \frac{1}{R}\partial_z \Psi \hat{\boldsymbol{R}} + \frac{\alpha\Psi}{R}\hat{\boldsymbol{\phi}}, \qquad \nabla\boldsymbol{\phi} = \frac{\hat{\phi}}{R}, \tag{7.28}$$

in which as in the previous section $\boldsymbol{R}, \phi, z$ are the standard cylindrical coordinates, $\hat{R}, \hat{\phi}, \hat{z}$ are the corresponding unit vectors, α is constant and $\Psi = \Psi(R, z)$ is an arbitrary function of R and z. The magnetic field can be calculated using (7.2) to be

$$\boldsymbol{B} = \frac{\alpha}{R}\partial_R \Psi \hat{z} - \frac{\alpha}{R}\partial_z \Psi \hat{\mathrm{R}} - \frac{\mathrm{D}^2\Psi}{R}\hat{\phi}, \tag{7.29}$$

in which according to Moffatt (1969) the operator D^2 is defined as

$$\mathrm{D}^2 = \partial_z^2 + R\partial_R \left(\frac{1}{R}\partial_R\right). \tag{7.30}$$

Obviously both $\boldsymbol{A}$ and $\boldsymbol{B}$ lie on the Ψ surfaces since

$$\nabla\Psi \cdot \boldsymbol{A} = \nabla\Psi \cdot \boldsymbol{B} = 0. \tag{7.31}$$

Let us define the variable r:

$$r = \sqrt{z^2 + (R - 1)^2}. \tag{7.32}$$

And let us assume that $\Psi = \Psi(r)$. In this case surfaces of constant Ψ are nested tori. The magnetic field is assumed to be confined between the tori $0 \leqslant r \leqslant a$ in which a is an arbitrary number such that $0 < a < 1$. A depiction of an R, z cross-section of the nested tori is given in figure 4. A typical field line of the magnetic field given in (7.29) is self-knotted in the sense of Moffatt (1969) as is evident from figure 5.

Next, following § 6.3 we define two functions with simple cuts ϕ^* and η^*, in which η^* can be considered as an angle varying over the small circle of the torus, while ϕ^*

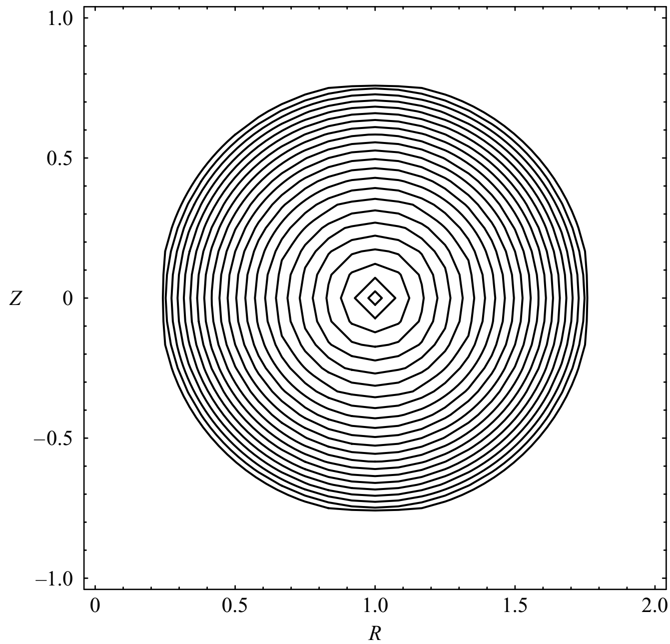

FIGURE 4. R, z cross-section of the nested tori.

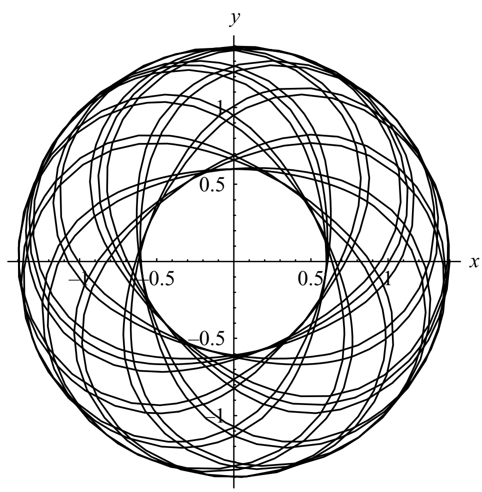

FIGURE 5. A numerically integrated field line assuming that $\Psi = r + r^3, \alpha = 1$ and starting from the point $R = 0.6, \phi = 0, z = 0$. The plot shows twenty rotations.

can be considered as an angle varying over the large circle of the torus. Hence $\phi^* = \phi$ is just the standard azimuthal angle and η^* can be defined as

$$\eta^* = \arctan \frac{z}{R-1}. \tag{7.33}$$

Obviously the Ψ surfaces are also the λ surfaces. Therefore we can calculate χ using (4.24) where we calculate the magnetic flux into the surface between the degenerate torus $r = 0$ and any other torus given by some value of Ψ. There are two ways to do

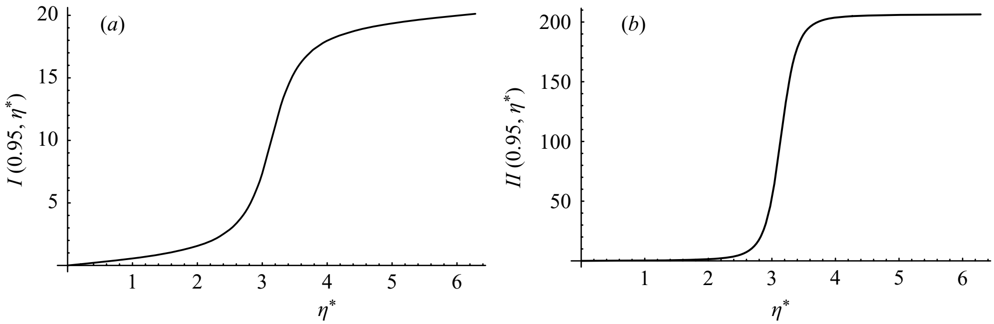

FIGURE 6. (*a*) $I(r, \eta^*)$ and (*b*) $II(r, \eta^*)$ for $r = 0.95$.

this but it seems that the simpler way is to take the surface that is perpendicular to $\hat{\eta}^*$ which is a unit vector in the η^*-direction. Hence we obtain

$$\chi = \frac{1}{2\pi}\int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S} = \frac{1}{2\pi}\oint \boldsymbol{A}\cdot \mathrm{d}l = \frac{1}{2\pi}\int_0^{2\pi} A_\phi R\,\mathrm{d}\phi = A_\phi R = \alpha\Psi. \tag{7.34}$$

In the above we assumed that $\Psi(0) = 0$. Let us now calculate the function η by solving (4.29). It is easy to show that η is of the form

$$\eta = \phi + C(z, R), \tag{7.35}$$

in which $C(z, R)$ is a solution of

$$B_\phi = \partial_z \chi\, \partial_R C - \partial_z \chi\, \partial_z C. \tag{7.36}$$

Writing the above equation in terms of r, η^* coordinates we obtain

$$\begin{gathered} -\frac{1}{1 + r\cos\eta^*}\left(\Psi'' + \frac{1}{1 + r\cos\eta^*}\frac{\Psi'}{r}\right) = -\frac{\alpha\Psi'}{r}\partial_{\eta^*} C, \\ \Psi' \equiv \frac{\mathrm{d}\Psi}{\mathrm{d}r}, \quad \Psi'' \equiv \frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2}. \end{gathered} \tag{7.37}$$

C can be integrated to yield the solution

$$C = \frac{1}{\alpha}\left[\frac{r\Psi''}{\Psi'} I(r, \eta^*) + II(r, \eta^*)\right], \tag{7.38}$$

in which

$$\begin{aligned} I(r, \eta^*) &\equiv \int \frac{\mathrm{d}\eta^*}{1 + r\cos\eta^*} \\ &= \frac{2}{\sqrt{1 - r^2}}\left[\arctan\left(\sqrt{\frac{1 - r}{1 + r}}\tan\left(\frac{\eta^*}{2}\right)\right) + \left\{\begin{array}{ll} 0, & 0 \leqslant \eta^* < \pi \\ \pi, & \pi \leqslant \eta^* < 2\pi. \end{array}\right.\right] \end{aligned} \tag{7.39}$$

and

$$II(r, \eta^*) \equiv \int \frac{\mathrm{d}\eta^*}{(1 + r\cos\eta^*)^2} = \frac{I(r, \eta^*)}{1 - r^2} - \frac{r\sin\eta^*}{(1 - r^2)(1 + r\cos\eta^*)}. \tag{7.40}$$

Plots of $I(r, \eta^*)$ and $II(r, \eta^*)$ are given in figures 6(*a*) and 6(*b*) respectively. Obviously $I(r, \eta^*)$ and $II(r, \eta^*)$ are non-single-valued functions. Their discontinuity values across the cut are given by

$$[I(r,\eta^*)] = \frac{2\pi}{\sqrt{1-r^2}}, \qquad [I(r,\eta^*)] = \frac{2\pi}{(1-r^2)^{3/2}}\ . \tag{7.41}$$

Therefore $C(r,\eta^*)$ is also a non-single-valued function. Using (7.38) we obtain the following discontinuity value of $C(r,\eta^*)$ across the cut:

$$[C] = \frac{2\pi}{\alpha\sqrt{1-r^2}}\left(\frac{r\Psi''}{\Psi'} + \frac{1}{1-r^2}\right). \tag{7.42}$$

It remains to calculate the value of the function ζ; this can be done using (7.5). Inserting into (7.5) the value of η given in (7.35), we obtain

$$\boldsymbol{A} = \chi\nabla\eta + \nabla\zeta = \frac{\alpha\Psi}{R}\hat{\phi} + \alpha\Psi\nabla C + \nabla\zeta. \tag{7.43}$$

Taking into account (7.28) in (7.43) leads to

$$\nabla\zeta = \frac{1}{R}\partial_R\Psi\hat{z} - \frac{1}{R}\partial_z\Psi\hat{R} - \alpha\Psi\nabla C. \tag{7.44}$$

The above equation implies that ζ is a function of R, z (or r, η^*) only. Writing (7.44) in terms of the r, η^* coordinates we arrive at a set of two equations:

$$\frac{1}{r}\partial_{\eta^*}\zeta = -\frac{\alpha\Psi}{r}\partial_{\eta^*}C + \frac{\Psi'}{1+r\cos\eta^*}, \qquad \partial_r\zeta = -\alpha\Psi\partial_r C. \tag{7.45}$$

Solving (7.45) we arrive at the solution

$$\zeta(r,\eta^*) = r\Psi' I(r,\eta^*) - \alpha\Psi C = rI(r,\eta^*)\left(\Psi' - \frac{\Psi\Psi''}{\Psi'}\right) - \Psi II(r,\eta^*). \tag{7.46}$$

Obviously $\zeta(r,\eta^*)$ is a non-single-valued function with the following discontinuity value across the cut:

$$[\zeta(r,\eta^*)] = \frac{2\pi}{\sqrt{1-r^2}}\left(r(\Psi' - \frac{\Psi\Psi''}{\Psi'}) - \frac{\Psi}{1-r^2}\right). \tag{7.47}$$

Let us calculate the magnetic helicity of the field using (7.11), (7.47) and (7.34); we arrive at the result

$$\begin{aligned}\mathscr{H}_M &= \int[\zeta]\mathrm{d}\Phi = \int_0^a [\zeta]2\pi\alpha\Psi'\,\mathrm{d}r \\ &= (2\pi)^2\alpha\int_0^a \frac{\mathrm{d}r}{\sqrt{1-r^2}}\left(r((\Psi')^2 - \Psi\Psi'') - \frac{\Psi\Psi'}{1-r^2}\right).\end{aligned} \tag{7.48}$$

A direct calculation using (7.1) will yield an identical result. This integral can be calculated either analytically or numerically for any reasonable function $\Psi(r)$. For example taking $\Psi(r) = r + r^3$ and $a = 0.9$ we calculated $\mathscr{H}_M$ numerically and obtained $\mathscr{H}_M = -4.6167(2\pi)^2\alpha$. To conclude: we do not see any principle difficulty in calculating the functions defined in this work for self-knotted magnetic field lines. It may be that the functions should be derived numerically if the nested tori are distorted.

7.4. *Cross-helicity conservation via the Noether theorem*

The conservation of helicity $\int \boldsymbol{v}\cdot\boldsymbol{\omega}\,\mathrm{d}^3x$ in ideal (non-magnetic) barotropic fluid when certain conditions are satisfied, in particular when $\boldsymbol{\omega}\cdot\boldsymbol{n} = 0$ on the (Lagrangian) surface bounding the volume of integration, was discovered by Moffatt (1969). Moreau (1977) has discussed the conservation of helicity from the group-theoretical point of view. In his paper he used an enlarged Arnold symmetry group Arnold (1966) of fluid element labelling to generate the conservation of helicity. Yahalom (1994, 1995, 1996) has shown that the symmetry group generating conservation of helicity becomes a very simple one-parameter translation group in the space of labels (alpha space) when represented by Lynden-Bell & Katz (1981) labelling. We will now show that in the case of magnetohydrodynamics the same one-parameter translation group will generate the magnetic cross-helicity via the Noether theorem.

Let us denote the initial position of a fluid element by (x_0^k), then by mass conservation

$$\rho(x^k)\mathrm{d}^3x = \rho\left(x_0^k\right)\mathrm{d}^3x_0 = \rho\left(x_0^k\right)\frac{\partial\left(x_0^1, x_0^2, x_0^3\right)}{\partial(\alpha^1, \alpha^2, \alpha^3)}\,\mathrm{d}^3\alpha. \tag{7.49}$$

Since the initial position of a fluid element cannot depend on time it must depend on the label only, and therefore by an appropriate choice of the α we obtain

$$\rho(x^k)\,\mathrm{d}^3x = \mathrm{d}^3\alpha, \qquad \rho = \frac{\partial(\alpha^1, \alpha^2, \alpha^3)}{\partial(x^1, x^2, x^3)}. \tag{7.50}$$

where we assume that the above expressions of course exist. Let us look at the action A defined in (2.6).

From the discussion following (2.15) we know that if the $\boldsymbol{\xi}$ variations disappear at times t_0, t_1 then A is extremal only if Euler's equations are satisfied and the boundary term disappears. If on the other hand we make a symmetry displacement, i.e. a displacement that makes δA vanish, and assume that Euler's equations are satisfied and the boundary term disappears, we obtain

$$\int_V \boldsymbol{v}\cdot\boldsymbol{\xi}\,\mathrm{d}^3\alpha = \text{const.} \tag{7.51}$$

This is Noether's theorem in its fluid mechanical form.

The α chosen so as to satisfy (7.50) are not unique, in fact one can always choose another set of variables, say $\tilde{\alpha}$ such that

$$\frac{\partial(\tilde{\alpha}^1, \tilde{\alpha}^2, \tilde{\alpha}^3)}{\partial(\alpha^1, \alpha^2, \alpha^3)} = 1. \tag{7.52}$$

It is quite clear that if the domain of integration is not modified any new set of α satisfying (7.52) can be chosen without affecting the value of the Lagrangian L. This is nothing but Arnold's (1966) alpha-space symmetry group under which L is invariant (see also Katz & Lynden-Bell 1985). For some flows the domain of integration can be modified without affecting L, and in that case we have additional elements in our symmetry group. If we make only small changes $\delta\alpha$ then we can define the group as follows:

$$\frac{\partial\delta\alpha_k}{\partial\alpha_k} = 0, \qquad \delta\boldsymbol{\alpha}\cdot\boldsymbol{n}|_{surface} = 0, \tag{7.53}$$

where $\boldsymbol{n}$ is a unit vector orthogonal to the surface of the alpha-space volume which we integrate over. The restriction $\delta\boldsymbol{\alpha}\cdot\boldsymbol{n}|_{surface} = 0$ is only needed when the infinitesimal transformation changes the domain of integration in such a way as to modify L. In this paper we are interested in the subgroup of translation, i.e.

$$\delta\alpha_k = a_k, \qquad a_k = const. \tag{7.54}$$

This subgroup of course does not satisfy $\delta\boldsymbol{\alpha}\cdot\boldsymbol{n}|_{surface}=0$ unless at least few of the α are cyclic or L is not affected by the modification of domain.

In § 4.3 we have defined the following three parameters: the magnetic load λ, the magnetic metage μ and η. Notice that since the magnetic lines are closed μ is an angular variable and we can translate it with out changing L. Choosing $\alpha^k=\chi, \eta, \mu$, and inserting those variables into (7.50) we re-derive (6.25).

The appropriate $\boldsymbol{\xi}$ symmetry displacement associated with the infinitesimal change in α_k is given by (6.29). For a metage displacement $\boldsymbol{\xi}$ takes the form

$$\boldsymbol{\xi} = -\frac{\partial \boldsymbol{r}}{\partial \mu}\delta\mu = -\delta\mu\frac{\boldsymbol{B}}{\rho}. \tag{7.55}$$

Inserting this expression into the boundary term in (2.15) will result in

$$\delta A_B = \int \mathrm{d}t \oint \mathrm{d}\boldsymbol{S}\cdot\left[\boldsymbol{B}\left(\tfrac{1}{2}\boldsymbol{v}^2 - w(\rho)\right) - \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{B})\right] = 0, \tag{7.56}$$

which is indeed Moffatt's condition for magnetic cross-helicity conservation (Moffatt 1969) as expected. Inserting (7.55) into (7.51) we obtain the conservation law

$$\int_V \boldsymbol{v}\cdot\frac{\partial \boldsymbol{r}}{\partial \mu}\mathrm{d}^3\alpha = \int_V \boldsymbol{v}\cdot\boldsymbol{B}\,\mathrm{d}^3x = \mathscr{H}_C. \tag{7.57}$$

Thus we conclude that the alpha translation group in the direction of μ generates conservation of helicity. (One could of course introduce the symmetry displacement $\boldsymbol{\xi}=\epsilon\boldsymbol{B}/\rho$; however, in this case one should show that the above displacement is a symmetry group displacement which is not obvious if we do not take into account Arnold's group and the Lynden-Bell & Katz labelling. Moreover in coordinate space the symmetry group appears arbitrary and complex depending on the flow considered as opposed to its apparent simplicity in alpha space).

8. Conclusion

In this paper we have reviewed variational principles for barotropic magnetohydrodynamics given by previous authors both in Lagrangian and Eulerian form. Furthermore, we have introduced our own Eulerian variational principles from which all the relevant equations of barotropic magnetohydrodynamics can be derived and which are in some sense simpler than those considered earlier. The variational principle was given in terms of six independent functions for non-stationary flows and three independent functions for stationary flows. This is less than the seven variables which appear in the standard equations of magnetohydrodynamics, which are the magnetic field $\boldsymbol{B}$ the velocity field $\boldsymbol{v}$ and the density ρ.

The equations in the non-stationary case have some resemblance to the equations deduced in a previous paper by Frenkel *et al.* (1982). However, in that previous work the equations were deduced from a postulated Hamiltonian. In the current work we show how this Hamiltonian can be obtained from our simplified Lagrangian using the canonical Hamiltonian formalism.

The appearance of a non-zero magnetic helicity and cross-helicity is connected with the fact that some of the functions which we defined are non-single-valued. This was elaborated to some extent in § 7 and was connected to the properties of the functions ζ, ν. We have also shown that the density of cross-helicity per unit of magnetic flux is also a conserved quantity and is equal to the discontinuity of ν. Furthermore, we have shown that the conservation of cross-helicity can be deduced using the Noether theorem from the symmetry group of magnetic metage translations.

It should be emphasized that for non-trivial topologies it is necessary to assume that some of the variables introduced in the non-stationary formalism are non-single-valued. That is, it is necessary to introduce a number of branch cuts in order to define single-valued branches of the field variables. In turn, these cuts along with the six field variables constitute an extended number of dynamic variables. The number of necessary cuts depends on the flow.

The problem of stability analysis and the description of numerical schemes using the described variational principles exceed the scope of this paper. We suspect that to achieve this we will need to add additional constants of motion constraints to the action as was done by Arnold (1965*a, b*); see also Yahalom, Katz & Inagaki (1994) hopefully this will be discussed in a future paper.

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